Linear Algebra

Lecture 01

Introduction to vector, Addition, Subtraction and Multiplication of Vectors

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Objectives

- Understand vectors vs. scalars.
- Represent vectors graphically and in components.
- Learn vector addition, subtraction, and multiplication methods

Outcomes

- Define and represent vectors.
- Add and subtract vectors (graphical & analytical).
- Find magnitude and direction of resultants.
- Calculate dot and cross products.
- Use vector math in problem-solving.

Vector : A vector is a set of numbers enclosed in a bracket.

Example 1

$$x = \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix}$$

Example 2

$$\mathbf{X} = \begin{bmatrix} 2 \\ 8 \end{bmatrix}$$

For this class: we will use cursive or bold font to denote a vector but most of the time we will use cursive notation

We denote the dimension of vectors with the notation
$\mathbf{x} \in R^2$
$x \in R^3$

The notation $x \in R^3$ says that the vector x will have three real numbers of **dimensions.**

Example 3

$$y = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$$

Example 4

$$\alpha = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \end{bmatrix}$$

Of course, a vector does not need to be denoted by a letter x it can be any letter like y or even Greek , α

A vector by itself does not tell as anything.

$$x = \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix}$$

We human should attach meaning to that numbers. For example these numbers can be

1. The number of point score by three people
2. The number of programming languages you have learned in the last three months

The possibilities are endless............

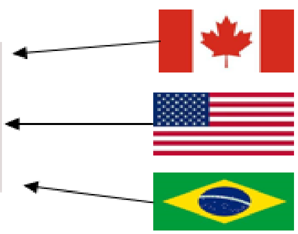
If we tell that the vector has the values of [1, 5, 3], it does not really mean anything to you.

Once meaning is given to a vector, it becomes data.

Giving meaning to vector.....

These are the scores of three soccer teams

$$x = \begin{bmatrix}1\\5\\3\end{bmatrix}$$



The three numbers are no longer just numbers they tell us a lot about the three teams

Once a vector become data we can. We can pose various queries(questions) about the information it holds. For example.....

1. What is the highest score?
2. What is the lowest score?
3. What is the average score?

We are going to call the three questions

q1, q2, q3

So once a vector becomes data we can use a queries to take the vector and return a meaningful value.

q1 using x ⟶ meaningful number

{here q1 will give us number 5}

We can translate the queries into mathematical language

q1 using x ---> meaningful number

- **Think of function like a query that take x and return a meaningful** value

q1(x) = some meaningful value

- **q1 here is equal to the max function** $q_1\left(\begin{bmatrix}1\\5\\3\end{bmatrix}\right) = \max(x) = 5$

- **This analogy extend to other query q2, q3** $q_2\left(\begin{bmatrix}1\\5\\3\end{bmatrix}\right) = \min(x) = 1$

Here we, establish that a function/query is just a way to extract information from vector

$q_3\left(\begin{bmatrix}1\\5\\3\end{bmatrix}\right) = \arg(x) = 3$

Vector addition

In linear algebra, vectors are often represented as column matrices or row matrices, and vector addition follows component-wise addition

For any **n-dimensional vectors**:

$$\mathbf{A} = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ \vdots \\ A_n \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ \vdots \\ B_n \end{bmatrix}$$

The sum of these vectors is:

$$\mathbf{R} = \mathbf{A} + \mathbf{B} = \begin{bmatrix} A_1 + B_1 \\ A_2 + B_2 \\ A_3 + B_3 \\ \vdots \\ A_n + B_n \end{bmatrix}$$

This equation works for **any dimension** n.

Vector subtraction

Vector subtraction follows the same rules as vector addition but involves **subtracting** corresponding components.

If

$$\mathbf{A} = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ \vdots \\ A_n \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ \vdots \\ B_n \end{bmatrix} \quad \mathbf{R} = \mathbf{A} - \mathbf{B} = \begin{bmatrix} A_1 - B_1 \\ A_2 - B_2 \\ A_3 - B_3 \\ \vdots \\ A_n - B_n \end{bmatrix}$$

In short: Subtract corresponding components of B from A.

Addition and subtraction of vectors

1) Vector addition

Example

$$x = \begin{bmatrix}1\\3\end{bmatrix}, y = \begin{bmatrix}2\\2\end{bmatrix}, then\ x + y = x + y = \begin{bmatrix}1\\3\end{bmatrix} + \begin{bmatrix}2\\2\end{bmatrix} = \begin{bmatrix}3\\5\end{bmatrix}$$

2) Vector subtraction

Example

$$x = \begin{bmatrix}1\\3\end{bmatrix}, y = \begin{bmatrix}2\\2\end{bmatrix}, then\ x - y = \begin{bmatrix}1\\3\end{bmatrix} - \begin{bmatrix}2\\2\end{bmatrix} = \begin{bmatrix}-1\\1\end{bmatrix}$$

Practice

Solve the following questions

1) If $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $y = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ then find R= $x + y$
2) If $x = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $y = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$ then find R= $x + y$
3) If $x = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$ and $y = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ then find R= $x + y$
4) If $x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $y = \begin{bmatrix} 3 \\ -2 \\ -4 \end{bmatrix}$ then find R= $x + y$
5) If $x = \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix}$ and $y = \begin{bmatrix} 3 \\ -2 \\ -4 \end{bmatrix}$ then find R= $x + y$
6) If $x = \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix}$, $y = \begin{bmatrix} 3 \\ -2 \\ -4 \end{bmatrix}$ and $w = \begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix}$ then find R= $x + y + w$

7) If $x = \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix}$, $y = \begin{bmatrix} 3 \\ -2 \\ -4 \end{bmatrix}$, $w = \begin{bmatrix} -5 \\ 2 \\ 3 \end{bmatrix}$ $and\ z = \begin{bmatrix} 5 \\ -2 \\ 3 \end{bmatrix}$

then find $x + y + w + z$

8) If $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $y = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ then find R= $x - y$

9) If $x = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $y = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$ then find R= $x - y$

10) If $x = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$ and $y = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$ then find R= $x - y$

11) If $x = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ and $y = \begin{bmatrix} -3 \\ -2 \\ -4 \end{bmatrix}$ then find R= $x - y$

12) If $x = \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix}$ and $y = \begin{bmatrix} 3 \\ -2 \\ -4 \end{bmatrix}$ then find R= $x - y$

The transpose operation

The transpose operation is when you flip the vector

Examples

$$x = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, then\ x^T = \begin{bmatrix} 1 & 3 \end{bmatrix}$$

$$y = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, then\ y^T = \begin{bmatrix} 1 & 2 & 2 \end{bmatrix}$$

$$z = \begin{bmatrix} 1 & 2 & 2 \end{bmatrix}, then\ z^T = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

Vector Multiplication

1) Hadamard Product

Example

$$x = \begin{bmatrix}1\\3\end{bmatrix}, y = \begin{bmatrix}2\\2\end{bmatrix} \; then \; x \odot y = \begin{bmatrix}1\\3\end{bmatrix} \odot \begin{bmatrix}2\\2\end{bmatrix} = \begin{bmatrix}2\\6\end{bmatrix}$$

2) Vector dot Product

For two vectors **A** and **B** in $\mathbb{R}^n$:

$$\mathbf{A} = \begin{bmatrix}a_1\\a_2\\\vdots\\a_n\end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix}b_1\\b_2\\\vdots\\b_n\end{bmatrix}$$

The **dot product** is calculated as:

$$\mathbf{A} \cdot \mathbf{B} = a_1b_1 + a_2b_2 + \cdots + a_nb_n = \sum_{i=1}^{n} a_ib_i$$

For example

$$\boldsymbol{x} = \begin{bmatrix} \mathbf{1} \\ \mathbf{3} \end{bmatrix}, \boldsymbol{y} = \begin{bmatrix} \mathbf{2} \\ \mathbf{2} \end{bmatrix} \boldsymbol{x} \cdot \boldsymbol{y} = \boldsymbol{x}^{\boldsymbol{T}} \boldsymbol{y} = [\mathbf{1} \quad \mathbf{3}] \begin{bmatrix} \mathbf{2} \\ \mathbf{2} \end{bmatrix} = \mathbf{1} \times \mathbf{2} + \mathbf{3} \times \mathbf{2} = \mathbf{8}$$

Practice 2

Find the dot product of the following vectors

1) 1) If $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$and $y = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ then find $x^T . y$
2) If $x = \begin{bmatrix} 3 \\ 4 \end{bmatrix}andy^T = [1 \quad 2]$ then find $x^T . y$
3) If $x = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$and $y = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ then find $x^T . y$
4) If $x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $y^T = [1 \quad 1 \quad 3]$ then find $x^T . y$
5) If $x^T = [-1 \quad -1 \quad 3]$and $y = \begin{bmatrix} 3 \\ -2 \\ -4 \end{bmatrix}$ then find $x^T . y$

3) Outer Product

Outer Product

$$\boldsymbol{x} = \begin{bmatrix} \mathbf{1} \\ \mathbf{3} \end{bmatrix}, \boldsymbol{y} = \begin{bmatrix} \mathbf{2} \\ \mathbf{2} \end{bmatrix} \ \boldsymbol{then}\ \boldsymbol{x} \otimes \boldsymbol{y} = \boldsymbol{x}\boldsymbol{y}^{\boldsymbol{T}} = \begin{bmatrix} \mathbf{1} \\ \mathbf{3} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{2} & \mathbf{2} \end{bmatrix} = \begin{bmatrix} \mathbf{2} & \mathbf{2} \\ \mathbf{6} & \mathbf{6} \end{bmatrix}$$

Vector multiplication by a constant value

Multiplication by a constant

$$\boldsymbol{\alpha} \begin{bmatrix} \boldsymbol{x}_{\mathbf{1}} \\ \boldsymbol{x}_{\mathbf{2}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\alpha} & \boldsymbol{x}_{\mathbf{1}} \\ \boldsymbol{\alpha} & \boldsymbol{x}_{\boldsymbol{l}} \end{bmatrix}$$

For example

$$\mathbf{3} \begin{bmatrix} \mathbf{1} \\ \mathbf{2} \end{bmatrix} = \begin{bmatrix} \mathbf{3} & \boldsymbol{x} & \mathbf{1} \\ \mathbf{3} & \boldsymbol{x} & \mathbf{2} \end{bmatrix} = \begin{bmatrix} \mathbf{3} \\ \mathbf{6} \end{bmatrix}$$