Linear Algebra

Lecture 11

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Objectives

- Understand and find unit vectors.
- Use the unit vector formula to get direction.
- Determine linear independence and matrix rank.
- Identify orthogonal vectors and normalize vectors.
- Compute eigenvalues, eigenvectors, and perform eigen decomposition.

Outcomes

- Define and find unit vectors.
- Write the formula for a unit vector.
- Test linear independence of vectors.
- Determine the rank of a matrix.
- Identify orthogonal vectors using the dot product.
- Normalize a vector to length 1.
- Compute eigenvalues and eigenvectors.
- Explain the basic idea of eigen decomposition

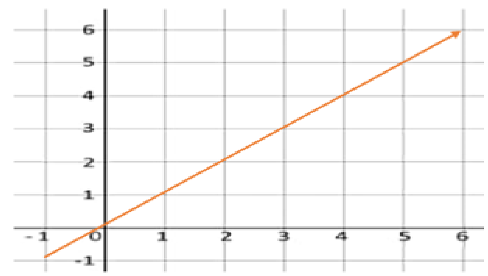
Linearly dependent and independence

Given 2 vectors

$$v_1 = \begin{bmatrix}1\\1\end{bmatrix}, \qquad v_2 = \begin{bmatrix}2\\2\end{bmatrix}$$

Their span would cover only 1 dimension

Linearly dependent

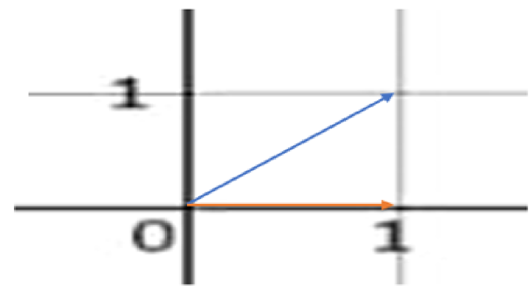


Given 2 vectors

$$v_1 = \begin{bmatrix}1\\0\end{bmatrix} v_2 = \begin{bmatrix}1\\1\end{bmatrix}$$

Their span would cover the entire 2D space

Linearly independent



The formal definition of linear independence

A set of vectors is linearly independent if and only if the equation:

$$c_1 v_1 + c_2 v_2 + \cdots c_k v_k = 0$$

Has only the trivial solution. What that mean is that these vectors are linearly independent when $c_1 = c_2 = \cdots = 0$ is the only possible solution to the vectors equation

If a set of vectors is not linearly independent we say that they are linearly dependent. Then we can write a linear dependence relation showing how one vector is combination of the other vectors.

Given this definition it implies that if we have a set of vectors

$v_1 v_2, v_3$ ... they are

Linearly independent : if their null space solution is the trivial solution.

Linearly dependent : if their null space solution is non-trivial solution.

Why linear independence?

Linear independence is a very important concept in data representation

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ 4 \\ 2 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 4 \\ 8 \\ 4 \\ 0 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 6 \\ 12 \\ 6 \\ 0 \end{bmatrix}, \quad v_5 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

We can represent the data as a matrix

$$A = \begin{bmatrix} 1 & 2 & 4 & 6 & 1 \\ 2 & 4 & 8 & 12 & 2 \\ 1 & 2 & 4 & 6 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We can assume the first row as basis and represent the data

$[1\ \ 2\ \ 4\ \ 6\ \ 1]$ become $[1v_1\ \ 2v_1\ \ 4v_1\ \ 6v_1\ \ 1v_1]$

Or we can use the first column as basis and represent the data as

$$\begin{bmatrix}1\\2\\1\\0\end{bmatrix} \quad \text{become} \quad X = \begin{bmatrix}1r_1\\2r_1\\1r_1\\0r_1\end{bmatrix}$$

Much compact way to represent data compare to matrix

Unit vectors are the preferred basis

We often represent data using independent basis

As a convention, we prefer to use **unite vectors.**

[A unite vector is a vector with length 1]

Examples of unite vectors

$$c = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \quad \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \\ 0 \end{bmatrix} \quad , \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{\frac{1}{2} + \frac{1}{2}} = \sqrt{\frac{2}{2}} = \sqrt{1} = 1$$

Example of not a unit vector

$$c = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\sqrt{(2)^2 + (1)^2} = \sqrt{4 + 1} = \sqrt{5} \neq 1$$

Equation for the unit vector

Example of NOT Unite Vector

$$C = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\sqrt{(2)^2 + (1)^2} = \sqrt{5} \neq 1$$

How do we turn this vector into a normal vector

Divide the vector by the length !!!

unite vector $\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \frac{1}{length}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2^2+1^2}}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$

You can double check your result the new length should be 1

Test for unit vector

$$\sqrt{\left(\frac{2}{\sqrt{5}}\right)^2 + \left(\frac{1}{\sqrt{5}}\right)^2} = \sqrt{\frac{4}{5} + \frac{1}{5}} = 1$$

Equation for the unit vector

The act of turning a vector into a unite vector is called **normalization**

Normalized vectors are normally denoted by hat

$u \rightarrow \hat{u}$ normalized

length$(u) \neq 1$

Length $(\hat{u}) = 1$

We refers to unite vector as **normalized vector**

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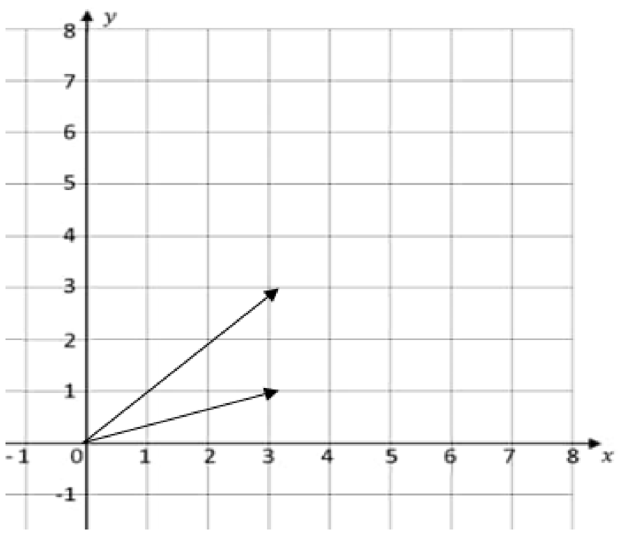
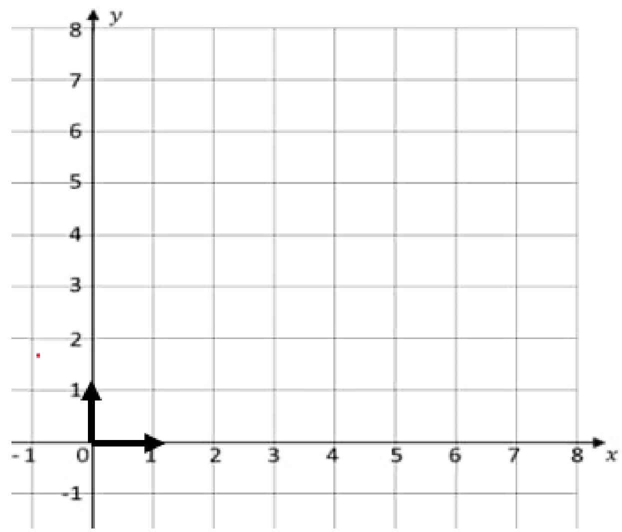
You can double check your result the new length should be 1

Besides having nice unite vector, we also want them to be perpendicular

In linear algebra the basis that are perpendicular are called orthogonal basis.





Let’s go over the definition of orthogonality

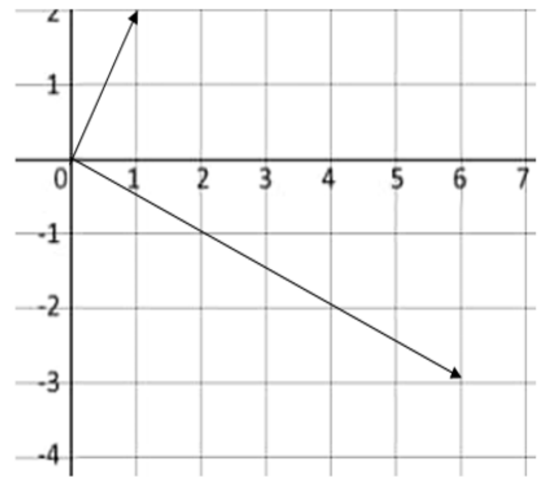
Orthogonality

Two vector $\vec{v}$ and $\vec{w}$ are called orthogonal if their do product is zero $\vec{v} \cdot \vec{w} = 0$

$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ *and* $\begin{bmatrix} 6 \\ -3 \end{bmatrix}$ are orthogonal in $\mathbb{R}^2$

In $\mathbb{R}^2$ vector are orthogonal if they are perpendicular to each other

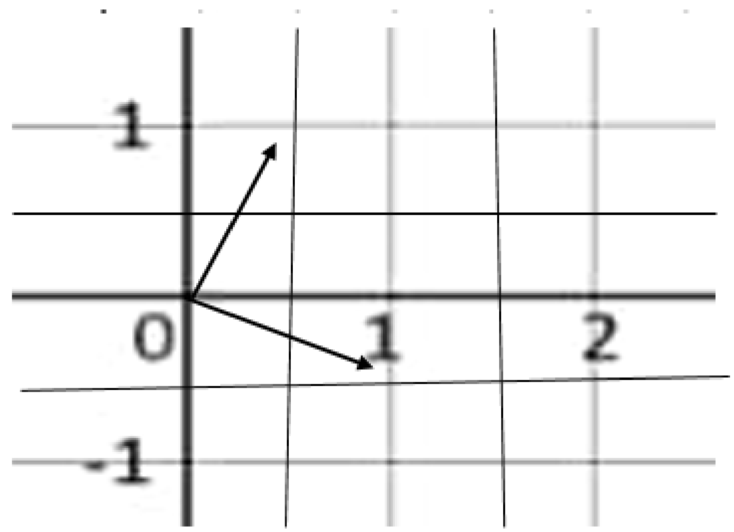
Taking the dot product and getting 0 tell you if the vectors are orthogonal.



Example of Orthonormal Pair

A set of vectors is called orthonormal if every vector in the set has length 1 (unit vector) and every pair of different vectors in the set is orthogonal (their dot product is zero).

$$\hat{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} \frac{6}{\sqrt{45}} \\ \frac{-3}{\sqrt{45}} \end{bmatrix}, \; v_1^T v_2 = 0$$



In general, we prefer orthonormal pairs as basis

$$\vec{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} \frac{6}{\sqrt{45}} \\ -\frac{3}{\sqrt{45}} \end{bmatrix} \xrightarrow[\text{As column vectors}]{} X = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{6}{\sqrt{45}} \\ \frac{2}{\sqrt{5}} & -\frac{3}{\sqrt{45}} \end{bmatrix}$$

We can identify linear independence immediately by multiplying X by itself transposed

$$X^T X = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{6}{\sqrt{45}} & -\frac{3}{\sqrt{45}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{6}{\sqrt{45}} \\ \frac{2}{\sqrt{5}} & -\frac{3}{\sqrt{45}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The matrix formed by orthonormal basis are called **orthonormal matrix**

If we get identity matrix the column vectors are automatically **independent**

Very important observation

Orthonormal vectors as basis make life easier

$$\vec{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} \frac{6}{\sqrt{45}} \\ -\frac{3}{\sqrt{45}} \end{bmatrix} \quad \xrightarrow[As\ column\ vectors]{} X = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{6}{\sqrt{45}} \\ \frac{2}{\sqrt{5}} & -\frac{3}{\sqrt{45}} \end{bmatrix}$$

This is very important we have previously seen

$$XX^{-1} = I$$

Therefore if $XX^{-1} = I$, then we can conclude that for orthonormal matrix

$$X^{-1} = X^T$$

This makes orthonormal matrices very special and easy to handle.

Norms

Norm: measures the size or "length" of vectors/matrices.

Generalized form of a vector p-norm:

$$\|X\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}$$

1-norm/L 1 norm/Manhattan norm:

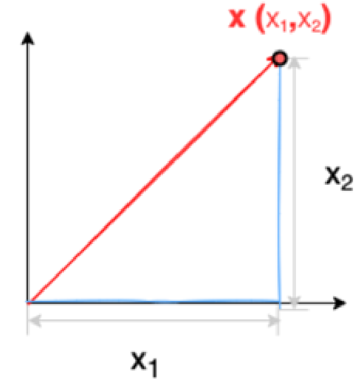
$$\|X\|_1 = \sum_{i=1}^{n} |x_i|$$

$$\|x\|_1 = |x_1| + |x_2|$$

L 2 norm is shortest distance from the origin.

2-norm/L 2 norm/Euclidean norm:

$$\|X\|_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2} \qquad \|x\|_2 = \sqrt{x_1^2 + x_2^2}$$



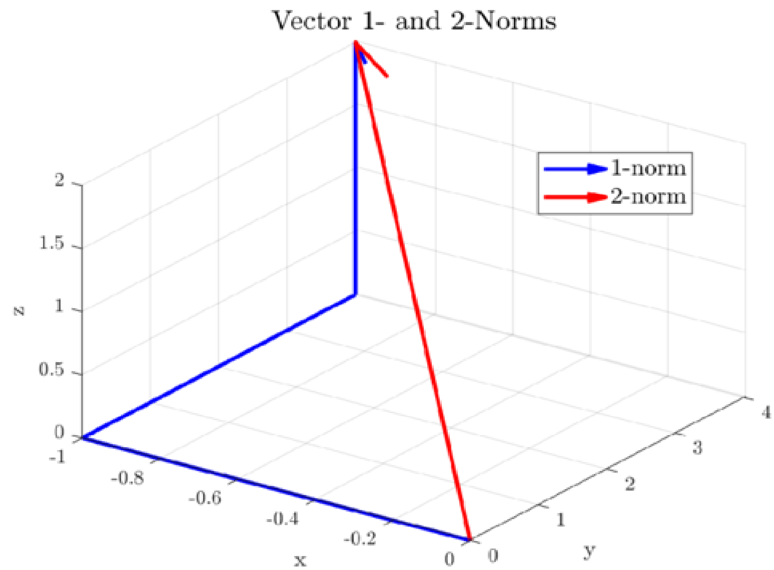
L 2 norm is shortest distance from the origin.

Example: Given $X = \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix}$, calculate the 1- norm and 2-norm.

- **1-norm:** $$\|X\|_1 = \sum_{i=1}^{n} |x_i| = 1 + 4 + 2 = \boxed{7}$$

- 2-norm:

$$\|X\|_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2} = \sqrt{(-1)^2 + 4^2 + 2^2} = \boxed{4.58}$$



Forbenius norm

$$\|\boldsymbol{A}\|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2}$$

The frobenius norm is the extension of L2 norm

Remember that given a vector $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ then the L2 norm is

$\|x\|_2 = \sqrt{2^2 + 1^2}$ The Frobenius Norm is the equivalent matrix version

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\|A\|_F = \sqrt{a_{11}^2 + a_{12}^2 + a_{21}^2 + a_{22}^2}$$

It means we square every element in the matrix and square root it sum

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EXAMPLE

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Then

$$\|A\|_F = \sqrt{1^2 + 2^2 + 1^2 + 3^2}$$

The symbol for determinant of A is $|A|$, the absolute value symbol

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{vmatrix} \mathrm{a} & \mathrm{b} & \mathrm{c} \\ \mathrm{d} & \mathrm{e} & \mathrm{f} \\ \mathrm{g} & \mathrm{h} & \mathrm{i} \end{vmatrix} = \mathrm{a}ei - \mathrm{afh} + b\,di - bgf + cdh - cge$$

Given a 2x2 and 3x3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{72} \end{bmatrix} \Rightarrow$$

The determinant of 2x2 matrix

$$|A| = a_{11} * a_{22} - a_{12^*}a_{21}$$

- The determinant of a 3x3 matrix is

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow$$

$$|A| = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} a_{23} \\ a_{31} a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

For matrices bigger than 3x3 we normally use computers to calculate their determinant.

But we should at least know the equations for 2x2 and 3x3 matrices.

Knowing the determinant is useful because $|A| = 0$ implies

- There is no unique solution.
- The matrix is singular.
- Inv(A) does not exist.
- Column vectors are linearly dependent.

For smaller matrices it is easier to find the determinant than Gaussian elimination.

The general formula for finding the det of any square matrix is

$$\det(A) = \sum(-1)^{i+j} a_{ij} \det(M_{ij})$$

Example: Find the determinant for each matrix

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 2 & 0 & -1 \\ 1 & 4 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 3 & 2 \\ -3 & -1 & -3 \\ 2 & 3 & 1 \end{bmatrix}$$

Solution

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\Rightarrow \quad |A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} a_{23} \\ a_{31} a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$det\begin{bmatrix} 2 & -3 & 1 \\ 2 & 0 & -1 \\ 1 & 4 & 5 \end{bmatrix} = 2\begin{vmatrix} 0 & -1 \\ 4 & 5 \end{vmatrix} - (-3)\begin{vmatrix} 2 & -1 \\ 1 & 5 \end{vmatrix} + 1\begin{vmatrix} 2 & 0 \\ 1 & 4 \end{vmatrix}$$

$$= 2[0-(-4)] + 3[10-(-1)] + 1[8-0]$$

$$= 8 + 33 + 8$$

$$det\begin{bmatrix} 1 & 3 & 2 \\ -3 & -1 & -3 \\ 2 & 3 & 1 \end{bmatrix}$$

$$= 1\begin{vmatrix} -1 & -3 \\ 3 & 1 \end{vmatrix} - 3\begin{vmatrix} -3 & -3 \\ 2 & 1 \end{vmatrix} + 2\begin{vmatrix} -3 & -1 \\ 2 & 3 \end{vmatrix}$$

$$= 1[-1-(-9)] - 3[-3-(-6)] + 2[-9-(-2)]$$

$$= 8 - 9 - 14 = -15$$

Previously we saw that matrix transformations vectors

- We have so far seen a lot of matrix transformation.
- $0x = 0$ like multiplying a number by 0
- $Ix = x$ like multiplying a number by 1
- $2Ix = 2x$ like multiplying a number by 2
- $R_\pi x = -x$ like rotating by π, 1→-1

All these transformation allow us to almost think of a x and matrix A just as number.

These transformation are from vectors perspective.

It tell us how a matrix can change a vector

Now we are going to look from the matrix perspectives.

Given a matrix what special number and vectors are there

- Special vectors: Eigen vectors
- Special number: Eigen value

Eigne values and eigne vectors are very special

So However, we have learned the concept of independence

If a set of bases is independent there is no redundancy

If a set of basis is dependent, there is redundancy

The existence of redundancy is what allow data compression

Eigenvalues and vectors allow to identify the minimum set of vectors that could span the same rank

For example, you could start off with 100 basis vectors. Once you have identified the eigenvectors and its associated eigenvalues, we might get

$$(v_1, \lambda_1 = 0.5), (v_2, \lambda_2 = 0.3)(v_3, \lambda_3 = 0.2), (v_4, \lambda_4 = 0.0001), (v_5, \lambda_5 = 0.0),$$

$$(v_6, \lambda_6 = 0.0), (\ldots, \lambda_{100} = 0.0)$$

The eigen values associated with the eigenvectors tells you the size of the vector in contributing to the span

Eigenvalues and vectors

Given a matrix A the eigenvectors of A are vectors that treat A as a number. Consider the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Notice that if you multiply it by 2 vectors it is equivalent to a number multiplying the vector

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

These vectors $v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$are\ special\ to\ A\ because\ they\ treat\ as\ a\ consntant\ multiplier\ of\ 1\ and\ 3$$

- **these vectors are called eigne vectors**
- **The multiplier values are called the eigne values**

The definition of eigenvalues and vectors is therefore

$$Av = \lambda v$$

Our goal today is to learn how to find these vectors and value.