

# Linear Algebra

## Lecture-12-13

Finding eigen value and eigen vector

**By Noorullah ibrahimi**

# Learning Objectives

By the end of studying Eigen Decomposition, a learner should be able to:

1. Define Eigenvalues and Eigenvectors
  - Understand what they represent and how they relate to linear transformations.
  - Find eigenvalues and corresponding eigenvectors.

## Learning outcomes

1. Define Eigenvalues and Eigenvectors
  - Understand the meaning of eigenvalues and eigenvectors.
  - Recognize how they relate to the equation  $Av = \lambda v$

Find eigenvalues and their eigenvectors

# Vector Norms

**Norm:** measures the size or “length” of vectors/matrices

- Generalized form of a vector  $p$ -norm:

$$\|X\|_p = \left( \sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}}$$

1-norm/ $L_1$  norm/Manhattan norm:

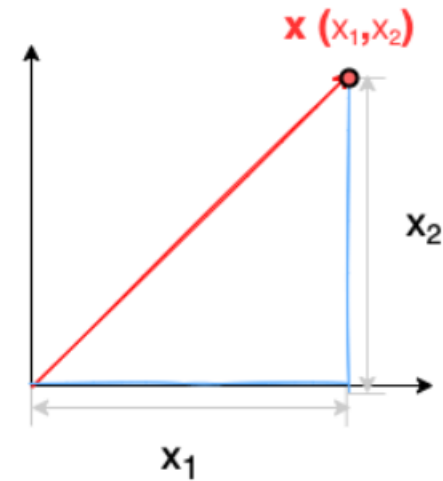
$$\|X\|_1 = \sum_{i=1}^n |x_i|$$

$$\|x\|_1 = |x_1| + |x_2|$$

2-norm/ $L_2$  norm/Euclidean norm:

$$\|X\|_2 = \sqrt{\sum_{i=1}^n |x_i|^2}$$

$$\|x\|_2 = \sqrt{x_1^2 + x_2^2}$$



- $L_2$  norm is shortest distance from the origin.

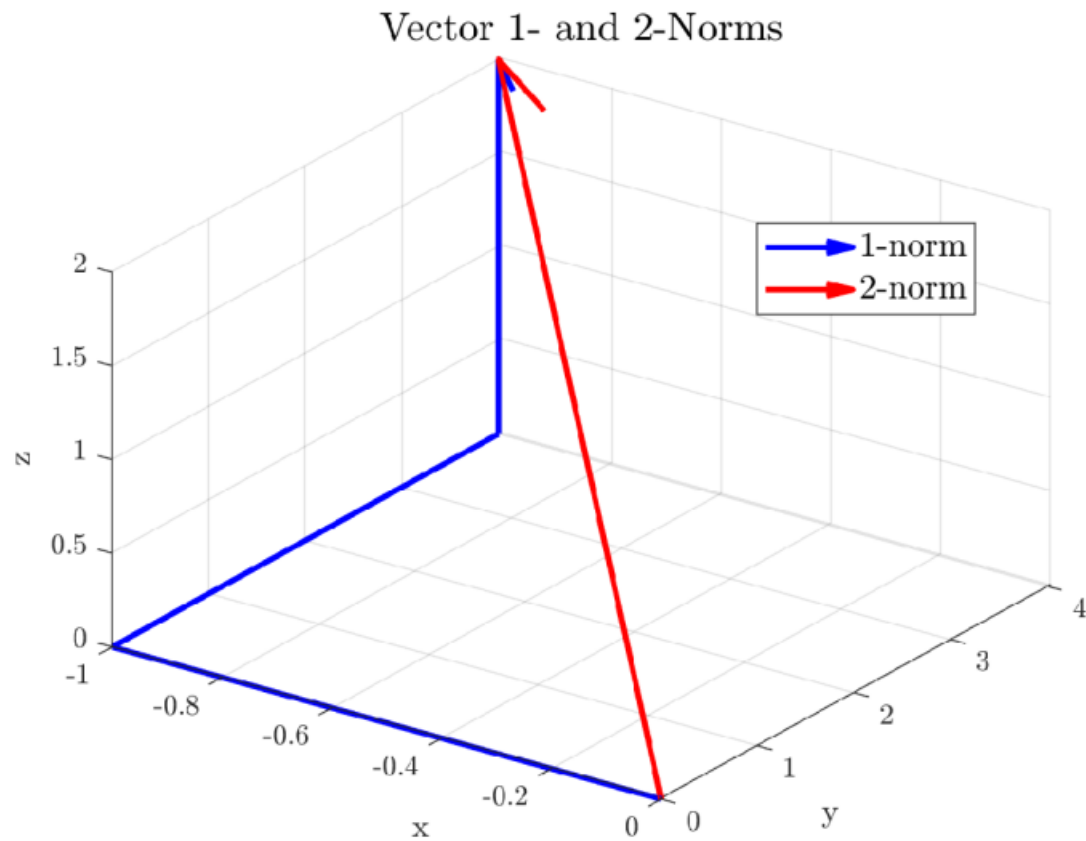
**Example:** Given  $X = \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix}$ , calculate the 1- norm and 2-norm.

- **1-norm:**

$$\|X\|_1 = \sum_{i=1}^n |x_i| = 1 + 4 + 2 = \boxed{7}$$

- **2-norm:**

$$\|X\|_2 = \sqrt{\sum_{i=1}^n |x_i|^2} = \sqrt{(-1)^2 + 4^2 + 2^2} = \boxed{4.58}$$



**Forbenius norm**

$$\|A\|_F = \sqrt{\sum_{i=1}^m \sum_{j=1}^n |a_{ij}|^2}$$

## The frobenius norm is the extension of L2 norm

Remember that given a vector  $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$  then the L2 norm is  $\|x\|_2 = \sqrt{2^2 + 1^2}$  The Frobenius Norm is the equivalent matrix version

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\|A\|_F = \sqrt{a_{11}^2 + a_{12}^2 + a_{21}^2 + a_{22}^2}$$

It means we square every element in the matrix and square root its sum

Example Given

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Then

$$\|A\|_F = \sqrt{1^2 + 2^2 + 1^2 + 3^2}$$

The symbol for determinant of A is  $|A|$  , the absolute value symbol

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei - afh + bdi - bgf + cdh - cge$$

**Given a 2x2 and 3x3 matrix**

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Rightarrow$$

The determinant of 2x2 matrix

$$|A| = a_{11} * a_{22} - a_{12} * a_{21}$$

The determinant of a 3x3 matrix is

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow$$

$$|A| = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

For matrices bigger than 3x3 we normally use computers to calculate their determinant

But we should at least know the equations for 2x2 and 3x3 matrices.

Knowing the determinant is useful because  $|A| = 0$  implies

1. There is no unique solution.
2. The matrix is singular.
3.  $\text{Inv}(A)$  does not exist.
4. Column vectors are linearly dependent.

For smaller matrices it is easier to find the determinant than Gaussian elimination.

**The general formula for finding the det of any square matrix is**

$$\det(A) = \sum (-1)^{i+j} a_{ij} \det(M_{ij})$$

**Example:** Find the determinant for each matrix

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 2 & 0 & -1 \\ 1 & 4 & 5 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 3 & 2 \\ -3 & -1 & -3 \\ 2 & 3 & 1 \end{bmatrix}$$

### Solution

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow |A| = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \Rightarrow$$

$$\begin{aligned} \det \begin{bmatrix} 2 & -3 & 1 \\ 2 & 0 & -1 \\ 1 & 4 & 5 \end{bmatrix} &= 2 \begin{vmatrix} 0 & -1 \\ 4 & 5 \end{vmatrix} - (-3) \begin{vmatrix} 2 & -1 \\ 1 & 5 \end{vmatrix} + 1 \begin{vmatrix} 2 & 0 \\ 1 & 4 \end{vmatrix} \\ &= 2[0 - (-4)] + 3[10 - (-1)] + 1[8 - 0] \\ &= 8 + 33 + 8 \end{aligned}$$

$$\begin{aligned} \det \begin{bmatrix} 1 & 3 & 2 \\ -3 & -1 & -3 \\ 2 & 3 & 1 \end{bmatrix} &= 1 \begin{vmatrix} -1 & -3 \\ 3 & 1 \end{vmatrix} - 3 \begin{vmatrix} -3 & -3 \\ 2 & 1 \end{vmatrix} + 2 \begin{vmatrix} -3 & -1 \\ 2 & 3 \end{vmatrix} \\ &= 1[-1 - (-9)] - 3[-3 - (-6)] + 2[-9 - (-2)] \\ &= 8 - 9 - 14 = -15 \end{aligned}$$

**Practice** find the determinant and measurement of the following matrices

$$1) \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

$$2) \begin{bmatrix} 3 & 0 \\ -1 & -2 \end{bmatrix}$$

$$3) \begin{bmatrix} -1 & 0 \\ 4 & -3 \end{bmatrix}$$

$$4) \begin{bmatrix} 5 & -3 \\ 7 & 4 \end{bmatrix}$$

$$5) \begin{bmatrix} 8 & 23 \\ 9 & 12 \end{bmatrix}$$

$$6) \begin{bmatrix} 0 & 2 \\ 5 & 6 \end{bmatrix}$$

$$7) \begin{bmatrix} 4 & -8 \\ 3 & -2 \end{bmatrix}$$

$$8) \begin{bmatrix} -2 & 5 \\ 6 & 3 \end{bmatrix}$$

$$9) \begin{bmatrix} 4 & 8 \\ 7 & -3 \end{bmatrix}$$

$$10) \begin{bmatrix} 1 & -1 & 0 \\ 2 & -2 & 1 \\ 4 & -3 & 3 \end{bmatrix}$$

$$11) \begin{bmatrix} 4 & 1 & 0 \\ 8 & 2 & 2 \\ 12 & 3 & -5 \end{bmatrix}$$

$$12) \begin{bmatrix} 0 & -11 & 5 \\ 3 & 0 & 1 \\ 5 & 3 & 4 \end{bmatrix}$$

$$13) \begin{bmatrix} 2 & 1 & 5 \\ 0 & 5 & 0 \\ 7 & 3 & 0 \end{bmatrix}$$

$$14) \begin{bmatrix} \frac{-1}{2} & \frac{3}{2} & \frac{3}{4} \\ 5 & -1 & 5 \\ 0 & 8 & 3 \end{bmatrix}$$

$$15) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$16) \begin{bmatrix} \frac{2}{3} & 5 & 0 \\ 2 & \frac{3}{2} & 2 \\ -1 & 6 & 1 \end{bmatrix}$$

## Previously we saw that matrix transformations vectors

- We have so far seen a lot of matrix transformation.
- $0x = 0$  like multiplying a number by 0
- $Ix = x$  like multiplying a number by 1
- $2Ix = 2x$  like multiplying a number by 2
- $R_{\pi}x = -x$  like rotating by  $\pi$ ,  $1 \rightarrow -1$

All these transformation allow us to almost think of a  $x$  and matrix  $A$  just as number.

These transformation are from vectors perspective.

- It tell us how a matrix can change a vector

Now we are going to look from the matrix perspectives.

- Given a matrix what special number and vectors are there

Special vectors: Eigen vectors

Special number: Eigen value

## Eigne values and eigne vectors are very special

So However, we have learned the concept of independence

If a set of bases is independent there is no redundancy

If a set of basis is dependent, there is redundancy

The existence of redundancy is what allow data compression

Eigenvalues and vectors allow to identify the minimum set of vectors that could span the same rank

For example, you could start off with 100 basis vectors. Once you have identified the eigenvectors and its associated eigenvalues, we might get

$$(v_1, \lambda_1 = 0.5), (v_2, \lambda_2 = 0.3), (v_3, \lambda_3 = 0.2), (v_4, \lambda_4 = 0.0001), (v_5, \lambda_5 = 0.0), \\ (v_6, \lambda_6 = 0.0), (\dots, \lambda_{100} = 0.0)$$

The eigen values associated with the eigenvectors tells you the size of the vector in contributing to the span

## Eigenvalues and vectors

Given a matrix A the eigenvectors of A are vectors that treat A as a number. Consider the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Notice that if you multiply it by 2 vectors it is equivalent to a number multiplying the vector

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

These vectors  $v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$  and  $v_2 =$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$  are special to A because they treat as a constant multiplier of 1 and 3

- these vectors are called eigen vectors
- The multiplier values are called the eigen values

- The definition of eigenvalues and vectors is therefore

$$Av = \lambda v$$

- Our goal today is to learn how to find these vectors and value

## How do we find the eigen values

If we follow the definition of the eigenvalues are eigenvectors we have

$$Av = \lambda v$$

This implies that we have

$$Av - \lambda Iv = 0$$

Since we have a 2x2 where matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}$$

Then the expression becomes

$$\left| \left( \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) v \right| = 0$$

$$\left| \begin{bmatrix} a_{11} - \lambda & a_{21} \\ a_{21} & a_{22} - \lambda \end{bmatrix} \right| = 0$$

$$(a_{11} - \lambda)(a_{22} - \lambda) - a_{12} * a_{21}$$

This expression is called the characteristic equation/ polynomial.

The eigenvalue is the solution of the characteristic polynomial.

### Example

Consider the matrix we previously saw

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

The characteristic equation is

$$|A - \lambda I| = 0$$

$$\left| \begin{bmatrix} 2 - \lambda & 1 \\ 1 & 2 - \lambda \end{bmatrix} \right| = (2 - \lambda)(2 - \lambda) - 1$$

$$\lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda - 1) = 0$$

From this, we can see that the solution for the characteristic equation is 3 and 1 they eigen values

Once we have the eigen values we can substitute it back in the equation let's first look at the case where the  $\lambda = 1$

$$A - \lambda I = 0$$

$$\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} v = 0$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} v = 0$$

This implies that the null space the solution for  $v$ , we know that

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \overrightarrow{e_1 - e_2} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

This implies that

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$$v_1 = -v_2$$

$$V = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -v_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} v_2$$

The space spanned by  $[-1 \ 1]$  is the null space and the eigenvector corresponding to  $\lambda = 1$ .

Consider the matrix we previously saw

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

The characteristic equation is

$$|A - \lambda I| = 0$$

$$\left| \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} \right| = (2-\lambda)(2-\lambda) - 1$$

$$\lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda - 1) = 0$$

From this, we can see that the solution for the characteristic equation is 3 and 1 they eigen values.

Once we have the eigen values we can substitute it back in the equation let's first look at the case where the  $\lambda = 3$

$$A - \lambda I = 0$$

$$\begin{bmatrix} 2-3 & 1 \\ 1 & 2-3 \end{bmatrix} v = 0$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} v = 0$$

This implies that the null space the solution for v, we know that

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \overrightarrow{e_1 + e_2} \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$$

This implies that

$$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$$-v_1 = -v_2$$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} v_2$$

The space spanned by  $\begin{bmatrix} 1 & 1 \end{bmatrix}$  is the null space and the eigenvector corresponding to  $\lambda = 3$ .

**Final Answers:**

**Eigenvalues:**

$$\lambda_1 = 1, \quad \lambda_2 = 3$$

**Corresponding Eigenvectors:**  $\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

**Obtain different eigenvectors**

Our eigenvalues and vectors for  $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$  are

$$\lambda_1 = 1, v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \text{ and } \lambda_2 = 3, v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Ad a standard when we are asked for the eigenvalue we present them in a sorted fashion (normally from the largest to the smallest )

So we have

$$\lambda = (3, 1)$$

$$v = [v_2, v_1] = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Notice that  $v_2$  is the first column. Lastly, notice that  $v_1$  and  $v_2$  are not the solutions; the solutions are the span of  $v_1$  and  $v_2$ .

**Example** Given the matrix

$$A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$$

- a) Find the eigenvalues of matrix  $A$ .
- b) Find the corresponding eigenvectors for each eigenvalue.

## SOLUTION

Step 1: Find the Eigenvalues

$$Av = \lambda v$$

If a matrix is multiplied with identity matrix, nothing will change

$$Iv = v$$

$$Av = \lambda Iv$$

$$(A - \lambda I)v = 0$$

We solve the characteristic equation:

$$\det(A - \lambda I) = 0 \quad \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 4 - \lambda & 1 \\ 2 & 3 - \lambda \end{bmatrix}$$

$$\det(A - \lambda I) = 0 \quad (4 - \lambda)(3 - \lambda) - (2)(1) = (4 - \lambda)(3 - \lambda) - 2$$

So,

$$\det(A - \lambda I) = \lambda^2 - 7\lambda + 12 - 2 = \lambda^2 - 7\lambda + 10$$

Set the determinant to zero:

$$\lambda^2 - 7\lambda + 10 = 0$$

Factor the quadratic:

$$(\lambda - 5)(\lambda - 2) = 0$$

**Eigenvalues:**

$$\lambda_1 = 5, \quad \lambda_2 = 2$$

**For  $\lambda=5$**

$$\text{Solve } (A - 5I) = 0 \quad \begin{bmatrix} 4 - 5 & 1 \\ 2 & 3 - 5 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix} [v] = 0$$

Solve:  $\begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

From the first equation:  $-v_1 + v_2 = 0 \Rightarrow v_2 = v_1$

So the eigenvectors corresponding to  $\lambda = 5$  is

$$v_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad v_1 \text{ not equal to } 0$$

**For  $\lambda=2$**

Solve  $(A-2I)=0$   $\begin{bmatrix} 4-2 & 1 \\ 2 & 3-2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$

Solve:  $\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

This reduces to:

$$2x + y = 0 \Rightarrow y = -2x$$

So the eigenvectors corresponding to  $\lambda=2$   $v_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

**Final Answers:**

**Eigenvalues:**

$$\lambda_1 = 5, \quad \lambda_2 = 2$$

**Corresponding Eigenvectors:**  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$  ,  $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

**Example**

Let 
$$\begin{bmatrix} 4 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

- a) Find the eigenvalues of matrix A.
- b) Find the eigenvectors corresponding to each eigenvalue.

**Solution:**

**Step 1: Find Eigenvalues**

Solve the characteristic equation:

$$\det(A - \lambda I) = 0$$

$$A - \lambda I = \begin{bmatrix} 4 - \lambda & 1 & 0 \\ 0 & 2 - \lambda & 0 \\ 0 & 0 & 3 - \lambda \end{bmatrix}$$

$$\det (A - \lambda I) = (4 - \lambda)(2 - \lambda)(3 - \lambda) = 0$$

So the eigenvalues are:

$$\lambda_1 = 4 \quad \lambda_2 = 2 \quad \lambda_3 = 3$$

Step 2: Find Eigenvectors

For  $\lambda=4$

Solve:  $(A - 4I)x = 0$

$$A - 4I = \begin{bmatrix} 4 - 4 & 1 & 0 \\ 0 & 2 - 4 & 0 \\ 0 & 0 & 3 - 4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

From the system:

- $x_2=0$
- $x_3=0$
- $x_1$  is free

So one eigenvector is:  $v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

For  $\lambda=2$

$$A - 2I = \begin{bmatrix} 4-2 & 1 & 0 \\ 0 & 2-2 & 0 \\ 0 & 0 & 3-2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solve:

- $2x_1 + x_2 = 0 \Rightarrow x_2 = -2x_1$
- $x_3 = 0$

So one eigenvector is:

$$v_2 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

For  $\lambda=3$

Solve:

- $x_2=0$
- $x_1$  is free
- $x_3$  is free

So one eigenvector is:

$$v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

**Final Answer:**

**Eigenvalues:**  $\lambda=4,2,3$

**Corresponding Eigenvectors:**

For  $\lambda = 4$  is  $v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

For  $\lambda = 2$  is  $v_2 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$

For  $\lambda = 3$  is  $v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Example

Let  $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

- 1) Find the Eigenvalues
- 2) Find the Eigenvectors for each eigenvalue

## Solution

### step 1: Find the Characteristic Polynomial

We solve:

$$\det(A - \lambda I) = 0$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 - \lambda & 0 & -1 \\ 1 & 2 - \lambda & 1 \\ 2 & 2 & 3 - \lambda \end{bmatrix}$$

$$\det(A - \lambda I) = 0 \Rightarrow \det \begin{bmatrix} 1 - \lambda & 0 & -1 \\ 1 & 2 - \lambda & 1 \\ 2 & 2 & 3 - \lambda \end{bmatrix}$$

or

$$(1 - \lambda) \begin{vmatrix} 2 - \lambda & 1 \\ 2 & 3 - \lambda \end{vmatrix} - 0 \begin{vmatrix} 1 & 1 \\ 2 & 3 - \lambda \end{vmatrix} + (-1) \begin{vmatrix} 1 & 2 - \lambda \\ 2 & 2 \end{vmatrix}$$

$$(1 - \lambda)[\lambda^2 - 5\lambda + 4] - 0 - 1[2 - 2(2 - \lambda)]$$

After few steps we get

$$-\lambda^3 + 6\lambda^2 - 11\lambda + 6 \dots \dots \dots (1)$$

This gives the characteristic polynomial:  $\lambda^3 - 6\lambda^2 + 11\lambda - 6$

## Step 2: Solve for Eigenvalues

Factoring:  $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = (\lambda - 1)(\lambda - 2)(\lambda - 3) = 0$

## How we can factor a cubic equation

Put  $\lambda = 1$  in equation 1

$$\text{So } 1 - 6 + 11 - 6 = 0$$

$\lambda = 1$  is root of equation (1)

so the remaining root of 1 are

$$\lambda^2 - 5\lambda + 6 = 0 \text{ or } (\lambda - 3)(\lambda - 2) = 0$$

$\lambda_2 = 3, \lambda_3 = 2$ , as 1 is also the root of 1 equation then all root of 1 are

$$, \lambda_2 = -1, \lambda_2 = 3, \lambda_3 = 2,$$

**Eigenvalues:**

$$\lambda_1=1 \quad \lambda_2=2 \quad \lambda_3=3$$

### Step 3: Find Eigenvectors

We solve  $(A - \lambda I)x$  for each eigenvalue:

**For  $\lambda=1$ :**

$$\text{Solve: } (A - 1I) = 0 \Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix} - 1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$$

$$\text{So now computing } (A - 1I)X = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Reduced Row Echelon Form (RREF) gives:

$$e1 \leftrightarrow e3 \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2e_1 + e_3 \rightarrow e_3 \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix}$$

$$\text{So } x_1 + x_2 + x_3 = 0$$

$$\text{And } -x_3 = 0 \text{ that mean } x_3 = 0$$

$$\text{And } x_1 + x_2 + x_3 = 0 \text{ putting the value of } x_3 = 0 \text{ in equation}$$

$$\text{We get } x_1 + x_2 + 0 = 0 \Rightarrow x_1 = -x_2$$

$$\text{Take } x_2 = 1 \Rightarrow x_1 = -1$$

$$\text{The corresponding eigen vector is } \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Find the eigen vectors corresponding to eigen values  $\lambda_2=2$   $\lambda_3=3$

**Example : find the Eigenvalues and Eigenvectors of a 3×3 Matrix**

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

**Step 1: Characteristic Polynomial**

$$\det(A - \lambda I) = \det \begin{bmatrix} 1 - \lambda & 0 & 0 \\ 0 & 2 - \lambda & 1 \\ 0 & 0 & 2 - \lambda \end{bmatrix}$$

This is upper triangular, so:

$$= (1 - \lambda)(2 - \lambda)^2 \Rightarrow \lambda_1 = 1, \quad \lambda_2 = 2 \text{ (with multiplicity 2)}$$

**◆ Step 2: Eigenvectors**

For  $\lambda = 1$ :

$$A - I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Solve:

$$\Rightarrow y + z = 0 \Rightarrow z = -y \Rightarrow x = \text{free}$$

Take  $y = 1, z = -1, x = 0$

$$\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

For  $\lambda = 2$ :

$$A - 2I = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow x = 0, z = 0, y = \text{free} \Rightarrow \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Also another vector from null space  $\rightarrow$  linearly independent one is:

$$\vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

**Final Answer:**

- Eigenvalues:  $\lambda = 1, 2$
- Eigenvectors:

- $\lambda = 1: \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$
- $\lambda = 2: \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

### EXAMPLE

Eigenvalues of a  $3 \times 3$  Matrix Find the eigenvalues of

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$$

**Solution** The characteristic polynomial of A is

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -4 & 17 & \lambda - 8 \end{bmatrix} = \lambda^3 - 8\lambda^2 + 17\lambda - 4$$

The eigenvalues of A must therefore satisfy the cubic equation

$$\lambda^3 - 8\lambda^2 + 17\lambda - 4 = 0$$

Dividing  $\lambda - 4$  into  $\lambda^3 - 8\lambda^2 + 17\lambda - 4 = 0$  show that we can be rewritten as  $(\lambda - 4)(\lambda^2 - 4\lambda + 1) = 0$

Thus, the remaining solutions of  $\lambda^3 - 8\lambda^2 + 17\lambda - 4$  satisfy the quadratic equation

$$\lambda^2 - 4\lambda + 1$$

which can be solved by the quadratic formula. Thus, the eigenvalues of A are

$$\lambda = 4, \quad \lambda = 2 + \sqrt{3}, \quad \text{and} \quad \lambda = 2 - \sqrt{3}$$

**Eigenvalues of an Upper Triangular Matrix** Find the eigenvalues of the upper triangular matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$$

**Solution**

Recalling that the determinant of a triangular matrix is the product of the entries on the main diagonal

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda - a_{11} & -a_{12} & -a_{13} & -a_{14} \\ 0 & \lambda - a_{22} & -a_{23} & -a_{24} \\ 0 & 0 & \lambda - a_{33} & -a_{34} \\ 0 & 0 & 0 & \lambda - a_{44} \end{bmatrix}$$

$$= (\lambda - a_{11})(\lambda - a_{22})(\lambda - a_{33})(\lambda - a_{44})$$

Thus, the characteristic equation is

$$(\lambda - a_{11})(\lambda - a_{22})(\lambda - a_{33})(\lambda - a_{44}) = 0$$

and the eigenvalues are

$\lambda = a_{11}, \lambda = a_{22}, \lambda = a_{33}, \lambda = a_{44}$  which are precisely the diagonal entries of A

### EXAMPLE

Eigenvalues of a Lower Triangular Matrix By inspection, the eigenvalues of the lower triangular matrix

$$A = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -1 & \frac{2}{3} & 0 \\ 5 & -8 & -\frac{1}{4} \end{bmatrix}$$

are  $\lambda = \frac{1}{2}$ ,  $\lambda = \frac{2}{3}$ , and  $\lambda = -\frac{1}{4}$ .

## Practice

1) Find Eigen vectors of the given matrix  $A$

$$A = \begin{bmatrix} -4 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

the matrix has the characteristics equation-

$$\begin{aligned} |\lambda I - A| &= \begin{vmatrix} \lambda + 4 & -1 & 0 \\ 0 & \lambda + 3 & -1 \\ 0 & 0 & \lambda + 2 \end{vmatrix} \\ &= (\lambda + 4)(\lambda + 3)(\lambda + 2) = 0 \end{aligned}$$

Find the corresponding eigen vectors to ,  $\lambda_1 = -4$  ,  $\lambda_2 = -3$ ,  $\lambda_3 = -2$

SOLUTION:

For  $\lambda_3 = -2$  the eigen vector is  $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

For ,  $\lambda_1 = -4$  ,  $\lambda_2 = -3$  the corresponding eigen vectors are  $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ ,  $\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$

2) Find the eigenvalues and corresponding eigenvectors of

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}$$

The characteristic polynomial is

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6$$

Its factors are

$$(\lambda - 1)(\lambda - 2)(\lambda - 3)$$

So the eigenvalues are 1, 2, and 3.

$$\mathbf{x}_1 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}$$

3) Find the eigenvalues and corresponding eigenvectors of

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 5 & 3 & 3 \end{bmatrix}$$

$$|\lambda I - A| = \begin{vmatrix} \lambda - 2 & 0 & 0 \\ 1 & \lambda - 1 & 0 \\ -5 & -3 & \lambda + 3 \end{vmatrix} = (\lambda - 2)(\lambda - 1)(\lambda + 3) = 0$$

$$\Rightarrow \lambda_1 = 2, \lambda_2 = 1, \lambda_3 = -3$$

Find the corresponding eigen vectors to each  $\lambda=1,2,-3$

4) Determine the characteristic root(Eigen values) and characteristic vectors(eigen vectors) of the matrix

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

the Eigen values of A are **For  $\lambda=0,3,15$**

find corresponding eigen vector **characteristic vectors**

5) Determine the Eigen values and eigen vectors of the matrix

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

the Eigen values of A are For  $\lambda=2,2,8$

find corresponding eigen vector **characteristic vectors**

6) Find the Eigen values and eigen vectors of the matrix

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$$

the Eigen values of A are For  $\lambda=2,3,5$

find corresponding eigen vector **characteristic vectors**

6) find the characteristic equation and eigenvalues

(a)  $\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

(b)  $\begin{bmatrix} -2 & -7 \\ 1 & 2 \end{bmatrix}$

(c)  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(d)  $\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$

(a)  $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

(b)  $\begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix}$

(c)  $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

(d)  $\begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix}$