Linear Algebra

Lecture-14-15

Eigen value Decomposition and singular value decomposition

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Learning Objectives of Eigen Decomposition

- Define eigenvalue decomposition: $A = v\,\Sigma v^{-1}$
- Diagonalize a matrix using eigenvectors and eigenvalues
- Use decomposition to compute powers of matrices: $A^K \ = v\,\Sigma^K\,v^{-1}$
- Apply to solve linear systems and differential equations
- Recognize when a matrix is diagonalizable (enough independent eigenvectors)
- Define SVD: $A = U\,\Sigma V^T$
- Identify I U, Σ and V^T
- Compute SVD for 2×2 and 3×3 matrices
- Use SVD to find rank, null space, and range

Learning outcomes

- Define eigenvalue decomposition: : $A = v\ \Sigma v^{-1}$
- Diagonalize a matrix using eigenvalues and eigenvectors
- Compute powers: $A^K\ = v\ \Sigma^K\ v^{-1}$
- Apply to solve linear systems and differential equations
- Identify when a matrix is diagonalizable (enough eigenvectors)
- Define SVD as $A = U\ \Sigma V^T$
- Identify matrices U, Σ and V^T
- Compute SVD for 2×2 and 3×3 matrices
- Use SVD to find rank, null space, and range of a matrix

Inverse properties

1. If $\boldsymbol{BA}$ return $\boldsymbol{I}$ then B must be inverse of $\boldsymbol{A^{-1}}$

$$\boldsymbol{BA = I \Rightarrow B = A^{-1}}$$

2. When we have the inverse of multiple matrices it has the properties

$$\boldsymbol{(ABC)^{-1} = C^{-1}B^{-1}A^{-1}}$$

3. Inverse of an inverse is just itself

$$\boldsymbol{(A^{-1})^{-1} = A}$$

4. The transpose of an inverse just the inverse of the transpose

$$\boldsymbol{(A^{-1})^{T} = (A^{T})^{-1}}$$

5. The inverse of a number α time A is

$$(\alpha A)^{-1} = \frac{1}{\alpha} A^{-1}$$

6. The inverse of a 2x2 matrix

$$A = \begin{bmatrix} a & b \\ C & d \end{bmatrix}^{-1} \Rightarrow \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Properties of Diagonal Matrices

- We previously learned about diagonal matrices. These are the matrices of all 0's except diagonal. For example

$$\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- A special case of diagonal matrix is identity matrix where the diagonal elements are all ones
- For diagonal matrices it is easy to take power of the matrix. We can just take the power or of each directly for example

$$\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1^2 & 0 \\ 0 & 3^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}$$

This even extends to the inverse

$$\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 1^{-1} & 0 \\ 0 & 3^{-1} \end{bmatrix} \Rightarrow \begin{bmatrix} 1^{-1} & 0 \\ 0 & 3^{-1} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

These properties are only for diagonal matrices and will not work for any regular matrix

- The transpose of a diagonal matrix is just itself

$$\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}^T$$

Quick practice

Try to find alternative expression for the following

Note: U and V are orthonormal basis. Implying that

- $U^T = U^{-1}$
- $V^T V = I$

Note: Σ is a diagonal matrix

$$(A+B+C)^T, (U^T V)^T, (U\Sigma V^T)^T, |U\Sigma V^T|^T, ((U^{-1}\Sigma V^T)^T)^{-1}$$

Quick practice solution

$$(A+B+C)^T = A^T + B^T + C^T$$

$$(U^T V)^T = V^T U$$

$$(U\Sigma V^T)^T = V\Sigma U^T$$

$$|U\Sigma V^T|^T = V\Sigma U^T$$

$$((U^{-1}\Sigma V^T)^T)^{-1} \quad = (V\Sigma (U^{-1})^T)^{-1}$$

$$= (V\Sigma (U^T)^{-1})^{-1} = U^T \Sigma^{-1} V^{-1}$$

$$= U^T \Sigma^{-1} V^{\mathrm{T}}$$

We previously learn to break down matrices via the eigen decomposition

A matrix A can be broken down into the multiplication of multiple matrices. This is called **Matrix Decomposition.** A very important decomposition is called the eigen value decomposition it state that

$$\boldsymbol{A = V\Sigma V^{-1}}$$

Where $\boldsymbol{V}$ consist of column eigen vectors

$$V = [v_1, v_2, v_3 \dots]$$

And Σ consist of a diagonal matrix of eigen values

$$\Sigma = \begin{bmatrix} \lambda_1 & 0 & 0 & \dots \\ 0 & \lambda_2 & 0 & \dots \\ 0 & 0 & \lambda_3 & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}$$

Knowing the eigen values makes the life easier in so many ways

- We can calculate the inverse of a matrix easily

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \qquad we\ can\ get\ this\ easily$$

- We can calculate the power of a matrix easily

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{1000} \quad we\ can\ get\ this\ easily$$

- It makes proving theoretical properties easier, we will see once we break down a matrix A into

$$\boldsymbol{A} = \boldsymbol{V\Sigma V^{-1}}$$

Having $\boldsymbol{V}$ as orthonormal matrix make life easier

$$\boldsymbol{A} = \boldsymbol{V\Sigma V^{T}}$$

Example: 2×2 Matrix

Let:

$$A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$$

Step 1: Find eigenvalues

$\det(A - \lambda I) = (4 - \lambda)(3 - \lambda) - (2)(1) = \lambda^2 - 7\lambda + 10 \Rightarrow \lambda = 5,\ 2$

Step 2: Eigenvectors

For $\lambda = 5$:

$$A - 5I = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \Rightarrow x = 2y \Rightarrow v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

For $\lambda = 2$:

$$A - 2I = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \Rightarrow x = -y \Rightarrow v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Step 3: Form v and $\boldsymbol{\Sigma}$

$$v = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}$$

Final Answer:

$$A = v\,\Sigma v^{-1}$$

Example: 2×2 Symmetric Matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Step 1: Eigenvalues

$$\det(A - \lambda I) = (\lambda - 3)(\lambda - 1) \Rightarrow \lambda = 3,\ 1$$

Step 2: Eigenvectors

For $\lambda = 3$:

$$A - 3I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \Rightarrow x = y \Rightarrow v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For $\lambda = 1$:

$$A - I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow x = -y \Rightarrow v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Step 3: v and

$$v = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

$$v^{-1}$$

For finding the $\boldsymbol{v^{-1}}$ $\boldsymbol{you\ very\ well}$

Final Answer:

$$A = v\ \Sigma v^{-1}$$

Example : 3×3 Diagonal Matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Already diagonal ⇒

$$v = I, \quad \Sigma = A$$

Final Answer:

$$A = v\,\Sigma v^{-1} = IAI^{-1}$$

Example

Given Matrix:

$$A = \begin{bmatrix} 6 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

From earlier:

- Eigenvalues: $\lambda_1 = 7, \lambda_2 = 2, \lambda_3 = 4$

Corresponding Eigenvectors:

We found:

- For $\lambda_1 = 7$: $v_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$
- For $\lambda_2 = 2$: $v_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$
- For $\lambda_3 = 4$: $v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Step 1: Construct $\boldsymbol{V}$

$$V = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Step 2: Construct $\boldsymbol{\Sigma}$ (Eigenvalues)

$$\Sigma = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Step 3: Compute V^{-1}

We compute the inverse of:

$$V = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This matrix is block diagonal in the third row/column, so we can invert the top-left 2×2 block and attach the 1 for the 3rd row/column.

Top-left 2×2 block:

$$\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

Determinant: $2(2) - (-1)(1) = 4 + 1 = 5$

Inverse:

$$\frac{1}{5}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$$

So full inverse:

$$V^{-1} = \begin{bmatrix} \frac{2}{5} & \frac{1}{5} & 0 \\ -\frac{1}{5} & \frac{2}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Final Decomposition

$$\boxed{A = V\Sigma V^{-1}}$$

Where:

$$V = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \quad V^{-1} = \begin{bmatrix} \frac{2}{5} & \frac{1}{5} & 0 \\ -\frac{1}{5} & \frac{2}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Practice

Show that for any two matrices A ,B and C of the same size ,verify all Properties of transpose and inverse matrices

$\boldsymbol{A} = \begin{bmatrix} \mathbf{1} & \mathbf{3} \\ \mathbf{2} & \mathbf{7} \end{bmatrix}$, B $= \begin{bmatrix} \mathbf{2} & \mathbf{3} \\ \mathbf{0} & \mathbf{8} \end{bmatrix}$ and C=$\begin{bmatrix} \mathbf{1} & \mathbf{3} \\ \mathbf{5} & \mathbf{20} \end{bmatrix}$

SVD (Singular Value Decomposition)

Eigen value decomposition is not the same as SVD(singular value decomposition) — but SVD is one type of matrix decomposition

SVD is one specific type of matrix decomposition.

It breaks any matrix A into three matrices:

$$A = U\Sigma V^T$$

where:

- U: orthogonal matrix (left singular vectors)
- Σ: diagonal matrix with **singular values**
- V: orthogonal matrix (right singular vectors)
-

Singular Value Decomposition (SVD) Overview

Given a real matrix A of size $m \times n$, the **SVD** is:

$$A = U\Sigma V^T$$

Where:

- U is an $m \times m$ **orthogonal matrix** (columns are left singular vectors),
- Σ is an $m \times n$ **diagonal matrix** (singular values on the diagonal),
- V is an $n \times n$ **orthogonal matrix** (columns are right singular vectors),
- V^T is the transpose of V.

Step-by-Step Procedure

Step 1: Compute $A^T A$ (size $n \times n$)

- This is a symmetric matrix.
- It helps find the **right singular vectors** (columns of V).

$$A^T A \Rightarrow \text{Eigenvalues of } A^T A = \sigma_i^2$$

Step 2: Find eigenvalues and eigenvectors of $A^T A$

1. Solve $\det(A^T A - \lambda I) = 0$ to find eigenvalues λ_i
2. Take square roots:

   $\sigma_i = \sqrt{\lambda_i}$ (these are the singular values)

3. Eigenvectors of $A^T A$ form the columns of V

Step 3: Build the matrix *Σ*

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_r \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \end{bmatrix}_{m\times n}$$

Step 4: Compute *U* using:

For each non-zero singular value σ_i:

$$u_i = \frac{1}{\sigma_i} A v_i$$

- Compute each left singular vector u_i using the right singular vector v_i
- Then complete U by choosing orthonormal vectors to fill the rest if needed

Step 5: Form the full decomposition

$$A = U\Sigma V^T$$

Where:

- U = matrix of left singular vectors
- Σ = diagonal matrix of singular values
- V = matrix of right singular vectors

Example

Let A= $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

SO the SVD is

$$A = U\Sigma V^T$$

Where:

- U: left singular vectors (orthogonal matrix)
- Σ: diagonal matrix with singular values($\sqrt{\lambda}$)
- V: right singular vectors (orthogonal matrix)

Compute $A^T A$

Since A is symmetric, $A = A^T$ so:

$$A^T A = A.A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$$

Find eigenvalues of $A^T A$

$$\det(A^T A - \lambda I) = \begin{bmatrix} 5-\lambda & 4 \\ 4 & 5-\lambda \end{bmatrix} = (5-\lambda)^2 - 16 \qquad \lambda = 9,1$$

Singular values are:

$$\sigma_1 = \sqrt{9}\ ,\ \sigma_2 = \sqrt{1}$$

Find right singular vectors (from $A^T A$)

For $\lambda = 9$:

$$(A^T A - 9I)x = \begin{bmatrix} -4 & 4 \\ 4 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For $\lambda = 1$:

$$(A^T A - 9I)x = \begin{bmatrix} -4 & 4 \\ 4 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Now normalize:

$$v_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad v_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

So,

$$V = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

Compute $U = AV\Sigma^{-1}$

Use:

$$u_1 = \frac{1}{\sigma_1} Av_1, \quad u_2 = \frac{1}{\sigma_2} Av_2$$

Let's compute:

For u_1

$$Av_1 = A \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \frac{3}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow u_1 = \frac{1}{3} \cdot \frac{3}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For u_2

$$Av_2 = A \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow u_2 = \frac{1}{1} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

So

$$U = [u_1 \quad u_2]$$

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

Final Answer (SVD of A):

$$A = U\Sigma V^T$$

Where:

$$\Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

$U = V$ because A is symmetric.

Symmetric matrix

A symmetric matrix is a matrix where $A^T = A$

For example,

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 4 \\ 3 & 4 & 0 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 4 \\ 3 & 4 & 0 \end{bmatrix}$$

This issue with eigen decomposition is that a symmetric matrix is necessary this imply that all of these nice theories and properties can not be apply to

- Tall matrices, for example

$$\begin{bmatrix} 1 & 3 \\ 2 & 0 \\ 1 & 0 \end{bmatrix} \neq \begin{bmatrix} 1 & 2 & 1 \\ 3 & 0 & 0 \end{bmatrix}$$

- Fat matrix

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ 1 & 0 \end{bmatrix}$$

- Any non symmetric matrix

$$\begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix} \neq \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$$

RESULTS

A is symmetric $\boldsymbol{U = V \ or} \quad \boldsymbol{U = V^{-1}}$

A is diagonal or orthogonal **Often** $\boldsymbol{U = V}$

A is general (non-symmetric**)** $\boldsymbol{U \neq V} \quad \boldsymbol{U \neq V^{-1}}$

When A is a symmetric matrix then

$$A = V\Sigma V^T = U\Sigma V^T$$

Powers are easy to find

If we have the singular value decomposition Again,

$$A = V\Sigma V^T$$

Then we can show how to get easily get the power of a matrix given that

$AAA = A^3$ then $AAA... = A^{1000}$

Would be very difficult to calculate. Eigne values five us a theoretical way to easily calculate them

$$(V\Sigma V^T)(V\Sigma V^T) = AA$$

$$V\Sigma\Sigma V^T =$$

$$V\Sigma^2V^T =$$

From this, we see that multiplying a matrix by itself twice, or A^2can be obtained by taking Σ^2. Again, since Σ is a diagonal matrix, it is easily to find the power.

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^2 = \begin{bmatrix} 1^2 & 0 \\ 0 & 2^2 \end{bmatrix}$$

Diagonal matrices are easy to find the power in general we have

$$A^n = V\Sigma^nV^T$$

Solving a linear system

From this observation it is very easy to solve linear system if we process the singular value decomposition for example

$$Ax = y$$

$$V\Sigma V^T x = y$$

$$(V\Sigma V^T)^{-1}(V\Sigma V^T)x = (V\Sigma V^T)^{-1}y$$

$$x = V\Sigma^{-1}V^T y$$

How does A^TA help

The idea of singular value decomposition or SVD is that while matrix A may not be a square matrix but $A^T A \ and \ AA^T$ are both square matrices

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ 1 & 0 \end{bmatrix}, \ AA^T = \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 3 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 3 & 9 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 10 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

We can now perform SVD on both $A^T A \ and \ AA^T$

The obvious question is, how does knowing the eigen decomposition of $A^T A \ and \ AA^T$ help with A itself

What does the eigen values and eigen vectors of $A^T A$ tell us about A ?

We start with

AS $A^T A$ is always a symmetric matrix

We can write

$$A^T A = V \Sigma V^T$$

Since Σ is diagonal matrix, we can split it

$A^T A = V\Sigma^{\frac{1}{2}}\Sigma^{\frac{1}{2}}V^T$ notice $\begin{bmatrix}1 & 0\\0 & 4\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}$

In between 2 Σs now we can put an identity matrix and I=$U^T U$

$$A^T\, A = V\Sigma^{\frac{1}{2}}I\Sigma^{\frac{1}{2}}V^T = V\Sigma^{\frac{1}{2}}U^T U\Sigma^{\frac{1}{2}}V^T$$

Lastly, remember if we have 2 orthonormal matrices, they multiply to be an identity matrix. For example, we come up with another orthonormal matrix called U, then $UU^T = I$ we will have

$$A^T A = V\Sigma^{\frac{1}{2}}U^T U\Sigma^{\frac{1}{2}}V^T = V\Sigma^{\frac{1}{2}}I\Sigma^{\frac{1}{2}}V^T$$

This is now a very telling equation because we know A can be decomposed into

$$A^T A = V\Sigma^{\frac{1}{2}}U^T U\Sigma^{\frac{1}{2}}V^T$$

$$A = U\Sigma^{\frac{1}{2}}V^T$$

Example

Let's use the matrix:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

This matrix is symmetric and has full rank. We will compute:

$$A = U\Sigma V^T$$

Step 1: Compute $A^T A$

$$A^T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \Rightarrow A^T A = A^T A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Now multiply:

$$A^T A = \begin{bmatrix} 1 \times 1 + 0 \times 0 + 1 \times 1 & 1 \times 0 + 0 \times 2 + 1 \times 0 & 1 \times 1 + 0 \times 0 + 1 \times 1 \\ 0 \times 1 + 2 \times 0 + 0 \times 1 & 0 \times 0 + 2 \times 2 + 0 \times 0 & 0 \times 1 + 2 \times 0 + 0 \times 1 \\ 1 \times 1 + 0 \times 0 + 1 \times 1 & 1 \times 0 + 0 \times 2 + 1 \times 0 & 1 \times 1 + 0 \times 0 + 1 \times 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

Find eigenvalues of $A^T A$

We solve the **characteristic equation**:

$$\det(A^T A - \lambda I) = 0$$

Let's write:

$$A^T A - \lambda I = \begin{bmatrix} 2-\lambda & 0 & 2 \\ 0 & 4-\lambda & 0 \\ 2 & 0 & 2-\lambda \end{bmatrix}$$

Use cofactor expansion along the second row (because of zeros):

$$\det = (4-\lambda) \cdot \det \begin{bmatrix} 2-\lambda & 2 \\ 2 & 2-\lambda \end{bmatrix}$$

Now compute the 2×2 determinant:

$$(2-\lambda)(2-\lambda) - (2)(2) = (2-\lambda)^2 - 4$$

So the full determinant becomes:

$$(4-\lambda)[(2-\lambda)^2 - 4] = 0$$

Let's solve:

$$(2-\lambda)^2 - 4 = 0 \Rightarrow (2-\lambda)^2 = 4 \Rightarrow 2-\lambda = \pm 2 \Rightarrow \lambda = 0,\ 4$$

So eigenvalues are:

- $\lambda_1 = 0$

- $\lambda_2 = 4$
- $\lambda_3 = 4$

Step 3: Singular Values

$$\sigma_i = \sqrt{\lambda_i} \Rightarrow \sigma_1 = \sqrt{4} = 2, \quad \sigma_2 = 2, \quad \sigma_3 = 0$$

So:

$$\Sigma = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Step 4: Find eigenvectors of $A^T A$ → Columns of V

We already know:

- Eigenvalue 4 (multiplicity 2)
- Eigenvalue 0 (multiplicity 1)

Let's find eigenvectors for $\lambda = 4$:

Solve:

$$(A^T A - 4I)x = 0 \Rightarrow \begin{bmatrix} -2 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & -2 \end{bmatrix} x = 0$$

This reduces to:

$$-2x_1 + 2x_3 = 0 \Rightarrow x_1 = x_3, \quad x_2 \text{ free}$$

So eigenvectors are:

- $\vec{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$
- $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Eigenvector for $\lambda = 0$ (null space):
Solve:

$$\begin{bmatrix} 2 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 2 \end{bmatrix} x = 0$$

This gives:

$$2x_1 + 2x_3 = 0 \Rightarrow x_1 = -x_3, \quad x_2 = 0 \Rightarrow \vec{v}_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

So final V is:

$$V = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

Step 5: Compute U

Use:

$$u_i = \frac{1}{\sigma_i} A\mathrm{v}_i$$

First vector:

$$Av_1 = A \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1+1 \\ 0+0 \\ 1+1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{2}} \\ 0 \\ \frac{2}{\sqrt{2}} \end{bmatrix} = \sqrt{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \mathrm{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Second vector:

$$A\mathrm{v}_2 = A \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \Rightarrow \mathrm{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Now we find third $\vec{u}_3$ orthogonal to both u_1 and u_2:
Let's pick:

$$u_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

So:

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Final Answer:

$$A = U\Sigma V^T$$

Where:

- $U = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \qquad \Sigma = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad V^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}$

SVD Example : 2×2 Matrix

Let:

$$A = \begin{bmatrix} 3 & 0 \\ 4 & 0 \end{bmatrix}$$

Step 1: Compute $A^T A$

$$A^T A = \begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 25 & 0 \\ 0 & 0 \end{bmatrix}$$

Step 2: Eigenvalues of $A^T A$

$$\text{Eigenvalues: } \lambda_1 = 25,\ \lambda_2 = 0 \Rightarrow \sigma_1 = \sqrt{25} = 5,\ \sigma_2 = 0$$

Step 3: Right singular vectors (columns of V)

From $A^T A$, eigenvectors:

- For λ = 25: $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

- For λ = 0: $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

So:

$$V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Step 4: Compute

$$Av_1 = A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \quad \| Av_1 \| = 5 \Rightarrow u_1 = \frac{1}{5}\begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Complete U to orthonormal set:

As u_1 and u_2 are orthonormal

So find a $u^T{}_1 . u_2 = 0$

$$u_2 = \begin{bmatrix} -4/5 \\ 3/5 \end{bmatrix}$$

$$U = \begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix}$$

Final SVD: our sigma2 is zero why like this

$$A = U\Sigma V^T = \begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example : 2×2 Matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Step 1: Compute $A^T A$

$$A^T A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow \lambda_1 = \lambda_2 = 2,\ \sigma = \sqrt{2}$$

Step 2: V from eigenvectors of $A^T A$

$$v_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad v_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow V = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Step 3: U from Av_i/σ_i

$$Av_1 = \begin{bmatrix} \frac{2}{\sqrt{2}} \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix} \Rightarrow u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$Av_2 = \begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix} \Rightarrow u_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Final SVD:

$$\Sigma = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}, \quad U = I, \quad V^T = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Example : 3×3 Diagonal Matrix

Let:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Already diagonal ⇒

- $\Sigma = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- $U = V = I$

Final SVD:

$$A = U\Sigma V^T = I \cdot \Sigma \cdot I^T$$

Example : 3×3 Identity Matrix

$$A = I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

SVD is trivial:

- $\Sigma = I, U = I, V = I$

Final SVD:

$$A = U\Sigma V^T = I \cdot I \cdot I^T$$

Example

Find Singular Value Decomposition (SVD) of the matrix:

$$A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$$

This is a 2×3 **non-square** matrix, so the SVD will be:

$$A = U\Sigma V^T$$

Where:

- $U \in \mathbb{R}^{2\times 2}$ (orthogonal matrix of left singular vectors),
- $\Sigma \in \mathbb{R}^{2\times 3}$ (diagonal matrix of singular values),
- $V \in \mathbb{R}^{3\times 3}$ (orthogonal matrix of right singular vectors).

Step 1: Compute AA^T and $A^T A$

These help us find U and V, respectively.

Compute AA^T:

$$AA^T = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 3\times 3 + 2\times 2 + 2\times 2 & 3\times 2 + 2\times 3 + 2\times(-2) \\ 2\times 3 + 3\times 2 + (-2)\times 2 & 2\times 2 + 3\times 3 + (-2)\times(-2) \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 9+4+4 & 6+6-4 \\ 6+6-4 & 4+9+4 \end{bmatrix} = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}$$

Step 2: Find eigenvalues of AA^T

Let $\det(AA^T - \lambda I) = 0 \Rightarrow \det\left(\begin{bmatrix} 17-\lambda & 8 \\ 8 & 17-\lambda \end{bmatrix}\right) = 0$ $(17-\lambda)^2 - 8^2 = 0$

$$\Rightarrow (17-\lambda)^2 = 64$$

$$\Rightarrow 17 - \lambda = \pm 8$$

$$\Rightarrow \lambda = 17 \pm 8 = 25,\ 9$$

So the **eigenvalues of AA^T** are:

- $\lambda_1 = 25 \Rightarrow \sigma_1 = \sqrt{25} = 5$
- $\lambda_2 = 9 \Rightarrow \sigma_2 = \sqrt{9} = 3$

Step 3: Find U

Solve for eigenvectors of $AA^T = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}$

For $\boldsymbol{\lambda = 25}$:

$$\left(\begin{bmatrix}17 & 8\\ 8 & 17\end{bmatrix} - 25I\right) = \begin{bmatrix}-8 & 8\\ 8 & -8\end{bmatrix} \Rightarrow \text{Solve: } -8x + 8y = 0 \Rightarrow x = y \Rightarrow \vec{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$$

For $\boldsymbol{\lambda = 9}$:

$$\left(\begin{bmatrix}17 & 8\\ 8 & 17\end{bmatrix} - 9I\right) = \begin{bmatrix}8 & 8\\ 8 & 8\end{bmatrix} \Rightarrow 8x + 8y = 0 \Rightarrow x = -y \Rightarrow \vec{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$$

So:

$$U = \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix}$$

Step 4: Compute V

Now find eigenvectors of $A^T A$ to get V

First compute $\boldsymbol{A^T A}$:

$$A^T A = \begin{bmatrix}3 & 2\\ 2 & 3\\ 2 & -2\end{bmatrix}\begin{bmatrix}3 & 2 & 2\\ 2 & 3 & -2\end{bmatrix} = \begin{bmatrix}9+4 & 6+6 & 6-4\\ 6+6 & 4+9 & 4-6\\ 6-4 & 4-6 & 4+4\end{bmatrix} = \begin{bmatrix}13 & 12 & 2\\ 12 & 13 & -2\\ 2 & -2 & 8\end{bmatrix}$$

Now find eigenvalues and eigenvectors of this 3×3 matrix. We already know the singular values are:

- $\sigma_1 = 5, \sigma_2 = 3$, so the eigenvalues of $A^T A$ must be $25,\ 9,\ 0$

We now find corresponding eigenvectors for these.

Let's just compute **one right singular vector** v_1 for $\sigma_1 = 5$ using:

$v_1 = \frac{1}{\sigma_1} A^T u_1 \Rightarrow u_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} A^T u_1 = A^T \cdot \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} = \frac{1}{\sqrt{2}}\left(A^T \begin{bmatrix}1\\1\end{bmatrix}\right) = \frac{1}{\sqrt{2}}\begin{bmatrix}3+2\\2+3\\2+(-2)\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}5\\5\\0\end{bmatrix} = \frac{5}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix} \Rightarrow v_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix}$

Similarly, find v_2 for $\sigma_2 = 3$ using $u_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$:

$$A^T u_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}3-2\\2-3\\2-(-2)\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\4\end{bmatrix} \Rightarrow v_2 = \frac{1}{\sqrt{18}}\begin{bmatrix}1\\-1\\4\end{bmatrix}$$

And v_3 is orthogonal to v_1, v_2 (can compute via Gram-Schmidt or cross-product method).

So,

$$V = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{18}} & \cdots \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{18}} & \cdots \\ 0 & \frac{4}{\sqrt{18}} & \cdots \end{bmatrix}$$

Final SVD form:

$$A = U\Sigma V^T$$

Where:

- $U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$
- $\Sigma = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}$
- V contains right singular vectors (as above)