Linear Algebra

Lecture-02

what is a matrix, Matrix Shapes and Names, How to transpose a matrix, Multiplying a matrix by a constant, Addition by a constant, Addition, Subtraction matrices

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Learning Objectives

- Understand vectors vs. scalars.
- Represent vectors graphically and in components.
- Learn vector addition, subtraction, and multiplication methods

Learning outcomes

- Define and represent vectors.
- Add and subtract vectors.
- Find magnitude and direction of resultants.
- Calculate dot and cross products.
- Use vector math in problem-solving

We previously learned about vectors

They look like this $x = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$

We can tell their dimension with this notation $x \in \mathbb{R}^3$

Today we are learning "what is a matrix?"

They look like this

$$X = \begin{bmatrix} 1 & 2 & 7 \\ 3 & 2 & 0 \end{bmatrix} \quad or \quad Y = \begin{bmatrix} 1 & 2 \\ 3 & 2 \\ 0 & 3 \end{bmatrix}$$

We can tell their dimension with this notation

$$X \in \mathbb{R}^{2\times 3} \quad or \quad Y \in \mathbb{R}^{3\times 2}$$

Matrices can take on any dimension

Notice that instead of a lowercase letters matrices are denoted with capital letters

Matrix Shapes and Names

A matrix that is wide is called a Fat matrix (with is greater than height). For example

$$\boldsymbol{X} = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \boldsymbol{o} & \mathbf{0} & \mathbf{1} \end{bmatrix}, \boldsymbol{Y} = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{0} & \mathbf{3} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} \end{bmatrix}, \mathbf{Z} = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} \end{bmatrix}$$

A matrix that is tall is called Tall matrix (width is less than height).

For example

$$\boldsymbol{X} = \begin{bmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{2} \\ \mathbf{0} & \mathbf{3} \end{bmatrix}, \boldsymbol{Y} = \begin{bmatrix} \mathbf{0} & \mathbf{3} \\ \mathbf{1} & \mathbf{1} \\ \mathbf{2} & \mathbf{1} \\ \mathbf{3} & \mathbf{1} \end{bmatrix}, \boldsymbol{Z} = \begin{bmatrix} \mathbf{1} & \mathbf{6} & \mathbf{10} \\ \mathbf{2} & \mathbf{7} & \mathbf{11} \\ \mathbf{3} & \mathbf{8} & \mathbf{12} \\ \mathbf{4} & \mathbf{9} & \mathbf{13} \end{bmatrix}$$

Square matrix

A matrix whose width is equal to height is called **Square matrix**.

For example

$$X = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}, \; Y = \begin{bmatrix} 1 & 4 & 8 \\ 2 & 5 & 9 \\ 3 & 6 & 10 \end{bmatrix}$$

Just like vectors we can also transpose matrices

The transpose operation is when you flop the vector down

- $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, then\ x^T = \begin{bmatrix} 1 & 2 \end{bmatrix}$
- $y = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, then\ y^T = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$
- $\mathrm{z} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}, then\ z^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

How to transpose a matrix

- $X = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}, \; then \; X^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$
- $Y = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}, then \; Y^T = \begin{bmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \end{bmatrix}$

Transpose of a transpose

X= $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}, thenX^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ $(X^T)^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

Practice 1

Find the transpose of the following matrices

1) $A = \begin{bmatrix} 1 & 2 & 5 \end{bmatrix}$

2) B= $\begin{bmatrix} 1 & 6 & 0 \\ -2 & 5 & -1 \end{bmatrix}$

3) $D = \begin{bmatrix} 1 & 0 & 5 & 3 \\ 2 & 3 & -3 & 4 \\ -1 & 3 & 0 & 3 \end{bmatrix}$

4) R= $\begin{bmatrix} 1 & 3 & 5 \\ 2 & -3 & 3 \\ 1 & 3 & -2 \\ 2 & 4 & 0 \end{bmatrix}$

5) E= $\begin{bmatrix} 3 & -1 & 3 \\ 4 & 3 & 0 \\ 5 & 4 & 5 \\ 7 & 5 & 6 \end{bmatrix}$

Addition &Multiplying a matrix by a constant

$$X = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \ then \quad 2X = \begin{bmatrix} 2\times 1 & 2\times 0 \\ 2\times 2 & 2\times 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 4 & 2 \end{bmatrix}$$

Addition by a constant

Matrix addition requires same size matrices (e.g., A+B only if A and B are both m×).

So mathematically:

A+c (where A is a matrix and c is a scalar) **is not valid in linear algebra**.

But in applied contexts (like programming or data science):

Many software tools (like MATLAB, NumPy, R) allow scalar + matrix by applying the scalar to each element of the matrix

$$X = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \ then\, X + 2 = \begin{bmatrix} 1+2 & 0+2 \\ 2+2 & 1+2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}$$

Matrix Addition

Definition:
Matrix addition is the operation of adding two matrices of the same order (same number of rows and columns) by adding their corresponding elements.

If

$$A = [a_{ij}] \quad \text{and} \quad B = [b_{ij}]$$

then the sum is

$$A + B = [a_{ij} + b_{ij}]$$

Example

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

Matrix Subtraction

Definition:
Matrix subtraction is the operation of subtracting two matrices of the same order by subtracting their corresponding elements.

If

$$A = [a_{ij}] \quad \text{and} \quad B = [b_{ij}]$$

then the difference is

$$A - B = [a_{ij} - b_{ij}]$$

Example:

$$A = \begin{bmatrix} 9 & 4 \\ 6 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 1 \\ 5 & 7 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 9-3 & 4-1 \\ 6-5 & 2-7 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 1 & -5 \end{bmatrix}$$

PRACTICE 2

Solve the following question

1) $A = [1 \quad 2 \quad 5]$ & B=$[-1 \quad 3 \quad 5]$ then find A+B?

2) B=$\begin{bmatrix} 1 & 6 & 0 \\ -2 & 5 & -1 \end{bmatrix}$ & $C = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 0 \end{bmatrix}$ then find A+C?

3) E=$\begin{bmatrix} 3 & -1 & 3 \\ 4 & 3 & 0 \\ 5 & 4 & 5 \\ 7 & 5 & 6 \end{bmatrix}$ &R=$\begin{bmatrix} 1 & 3 & 5 \\ 2 & -3 & 3 \\ 1 & 3 & -2 \\ 2 & 4 & 0 \end{bmatrix}$ then find R-E & R+E?

4) $D = \begin{bmatrix} 1 & 0 & 5 & 3 \\ 2 & 3 & -3 & 4 \\ -1 & 3 & 0 & 3 \end{bmatrix}$& $F = \begin{bmatrix} 1 & -1 & 0 & 3 \\ 3 & 3 & 7 & -1 \\ 5 & 6 & 5 & 5 \end{bmatrix}$ then find D+F & D-F?

Matrix Hadamard Product

Examples

$$X = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, Z = \begin{bmatrix} 1 & 3 \\ 2 & 3 \end{bmatrix} \text{ then } X \odot Z = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 3 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4 & 3 \end{bmatrix}$$

$$Y = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, Z = \begin{bmatrix} 1 & 3 \\ 2 & 3 \end{bmatrix} \text{ then } Y \odot Z = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \odot \begin{bmatrix} 1 & 3 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1a & 3b \\ 2c & 3d \end{bmatrix}$$

Hadamard product can be performed on any two vectors(matrices) of the same dimension.

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 3 \end{bmatrix} B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} A \odot B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 3 \end{bmatrix} \odot \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$2A \odot B = 2 \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 3 \end{bmatrix} \odot \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 4 & 2 \\ 0 & 6 \end{bmatrix} \odot \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}$$

Matrix Dot Product

We pair up every combinations of vectors with each other.

Matrix dot product is the default multiplication. So dot is not necessary. It is always often omitted.

$$X = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, Z = \begin{bmatrix} 1 & 3 \\ 2 & 3 \end{bmatrix} \; then \; XZ = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} [1 \;\; 0] \begin{bmatrix} 1 \\ 2 \end{bmatrix} & [1 \;\; 0] \begin{bmatrix} 3 \\ 3 \end{bmatrix} \\ [2 \;\; 1] \begin{bmatrix} 1 \\ 2 \end{bmatrix} & [2 \;\; 1] \begin{bmatrix} 3 \\ 3 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 1*1+0*2 & 1*3+0*3 \\ 2*1+1*2 & 2*3+1*3 \end{bmatrix}$$

Unlike the Hadamard product the size does not have to be the same.

Instead, the number of columns on the left must be the same as the number of rows on the right

$$X = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}, Z = \begin{bmatrix} 1 & 3 \\ 2 & 3 \end{bmatrix} \; then \; XZ = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 4 & 9 \\ 2 & 3 \end{bmatrix}$$

The transposed of two matrices multiplied together is the transpose of each individual matrix switched in location

$$(XY)^T = Y^T X^T$$

Practice 3

Find the dot product of following vectors and matrices

1) If $X = \begin{bmatrix} 1 & 2 \end{bmatrix} \& Y = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ find $X.Y \&\ \ Y.X$
2) If $x = \begin{bmatrix} 1 & 2 & 4 \end{bmatrix}\ \&\ y = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$ find $x.\,y\ \&\ y.\,x$
3) If $y = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} \&\ z = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ find $y.\,z$
4) If $x = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 0 & -1 \end{bmatrix}\ \&\ y = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ find $x.\,y$
5) If $x = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 0 & -1 \end{bmatrix} \& y = \begin{bmatrix} 2 & 0 \\ 3 & -1 \\ 5 & 3 \end{bmatrix}$ find $x.\,y$
6) If $x = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \\ 2 & 3 & 0 \end{bmatrix} \& y = \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 5 & 0 \end{bmatrix}$ find $x.\,y$

7) If $A = \begin{bmatrix} 2 & 3 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ & $B = \begin{bmatrix} 1 & 0 & -2 \\ -1 & 3 & 3 \\ 0 & 2 & 1 \end{bmatrix}$ find A.B

8) If $A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$ & $B = \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 4 & 3 & 4 \\ 1 & 0 & 0 & 2 \end{bmatrix}$ find A.B

9) If $A = \begin{bmatrix} 3 & 1 & 1 & 1 \\ 0 & 0 & 3 & 0 \\ 0 & 1 & 4 & 1 \end{bmatrix}$ & $B = \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 4 & 3 & 4 \\ 1 & 2 & -1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ find A.B

10) If $B = \begin{bmatrix} 3 & 1 & 1 & 1 \\ 0 & 0 & 3 & 0 \\ 1 & 2 & 1 & 0 \\ 2 & 3 & 2 & 2 \end{bmatrix}$ & $C = \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 4 & 3 & 4 \\ 1 & 2 & -1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ find B.C

11) If $D = \begin{bmatrix} 3 & 1 & 1 & 1 & 1 \\ 0 & 0 & 3 & 0 & -2 \\ 1 & 2 & 1 & 3 & 0 \\ 2 & 3 & 2 & 5 & 1 \end{bmatrix}$ & $E = \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 4 & 3 & 4 \\ 1 & 2 & -1 & 0 \\ 1 & 0 & 2 & 1 \\ 2 & 1 & 0 & 0 \end{bmatrix}$ find B.C

The order of matrix dot product cannot be flipped.

$$IF\ X = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} and \quad Y = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

then

$$XY = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$$

$$YX = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix}$$

$$XY \neq YX$$