Linear Algebra

Lecture-04

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Learning Objectives

- **Understand the method** of solving systems of linear equations step by step using row operations.
- **Apply row operations** (swap, multiply, add/subtract rows) to simplify matrices.
- **Transform a system** into **upper triangular form** (row echelon form).
- **Use back substitution** to find the solution of the system.

Learning outcomes

- **Form an augmented matrix** from a system of linear equations.
- **Perform row operations** (swap, multiply, add/subtract rows).
- **Reduce a matrix to row echelon form** (upper triangular form).
- **Apply back substitution** to find unknown variables.
- **Solve 2×2, 3×3, and larger systems** of linear equations systematically.

Solving a system of equations

Suppose you have a system of linear equations:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

Step 1: Write the augmented matrix

$$Example: \boldsymbol{Ax} = \boldsymbol{b} \quad \rightarrow [A \mid b]$$

1) **Forward elimination** (make zeros below pivots):

- Choose a **pivot element** (non-zero entry in a column).
- Use **row operations** to make all entries below the pivot zero.

2) Allowed operations:

- Swap two rows
- Multiply a row by a nonzero constant
- Add/subtract multiples of one row to another

3) Continue for each column until you have an **upper triangular matrix** (all zeros below the main diagonal).

4) Back substitution:

- Solve the equations starting from the last row upward.

Example 1 : solve the system of linear equations using Gaussian law

$$\begin{aligned} x + 2y &= 8 \\ 3x - y &= 7 \end{aligned} \quad \Longrightarrow \quad \left[\begin{array}{cc|c} 1 & 2 & 8 \\ 3 & -1 & 7 \end{array}\right]$$

$-3e_{1+}e_2 \rightarrow e_2$

$$\left[\begin{array}{cc|c} 1 & 2 & 8 \\ 0 & -7 & -17 \end{array}\right]$$

Solve $-7y = -17 \Rightarrow y = \frac{17}{7}$

Back-substitute: $x = 8 - 2y = 8 - \frac{34}{7} = \frac{22}{7}$

So vector $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{22}{7} \\ \frac{17}{7} \end{bmatrix}$ **will satisfy the system**

Practice 1

Solve the systemS of linear equations using Gaussian elimination law

1)

$$2x + y = 5$$
$$x - y = 1$$

2)

$$4x - 3y = 1$$
$$2x + y = 7$$

3)

$$5x + 2y = 12$$
$$x - 4y = -6$$

4)

$$3x + 2y = 10$$
$$6x - y = 7$$

Example

Solve the system of linear equations using Gaussian elimination low

$$\begin{aligned} x + y + z &= 6 \\ 2x - y + 3z &= 14 \\ 3x + 4y - 2z &= 3 \end{aligned} \quad \Longrightarrow \quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & -1 & 3 & 14 \\ 3 & 4 & -2 & 3 \end{array}\right]$$

$-2e_{1+}e_2 \to e_2$

$-3e_{1+}e_3 \to e_3$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -3 & 1 & 2 \\ 0 & 1 & -5 & -15 \end{array}\right]$$

Now

Eliminate below the pivot in col 2:

$\frac{1}{3}e_{2+}e_3 \longrightarrow e_3$

From last equation :

$$-\frac{14}{3}z = -\frac{43}{3} \Rightarrow z = \frac{43}{14}$$

From 2nd equation

$$-3y + z = 2 \Rightarrow -3y + \frac{43}{14} = 2 \Rightarrow y = \frac{5}{14}$$

From 1st equation

$$x + y + z = 6 \Rightarrow x = 6 - \frac{5}{14} - \frac{43}{14} = \frac{18}{7}$$

So the vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{18}{7} \\ \frac{5}{14} \\ \frac{43}{14} \end{bmatrix}$ will satisfy the system

Practice 2

Solve the systems of linear equations using Gaussian elimination law

1)

$$2x + 3y - z = 5$$
$$4x - y + 2z = 10$$
$$-2x + y + z = -1$$

2)

$$x + 2y + 3z = 14$$
$$2x - y + z = 2$$
$$3x + y - z = 5$$

3)

$$x + 2y + 3z = 14$$
$$2x - y + z = 2$$
$$3x + y - z = 5$$

4)

$$2x + y + z = 4$$
$$x - y + 2z = 6$$
$$3x + 2y - z = 7$$

5)

$$x - y + z = 1$$
$$2x + 3y - z = 4$$
$$4x + y + 2z = 10$$

Now let's solve a system of four equations.

$$\begin{aligned} 2x_1 + 4x_2 + 2x3 + x_4 &= 8 \\ 3x_1 + 1x_2 + 1x_3 + 0x_4 &= 4 \\ 0x_1 + 2x_2 + 2x_3 + 0x_4 &= 2 \\ 1x_1 + 0x_2 + 0x_3 + 1x_4 &= 3 \end{aligned}$$

To augmented matrix $\longrightarrow$

$$\left[\begin{array}{cccc|c} 2 & 4 & 2 & 1 & 8 \\ 3 & 1 & 1 & 0 & 4 \\ 0 & 2 & 2 & 0 & 2 \\ 1 & 0 & 0 & 1 & 3 \end{array}\right]$$

Final goal $\longrightarrow$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & - \\ 0 & 1 & 0 & 0 & - \\ 0 & 0 & 1 & 0 & - \\ 0 & 0 & 0 & 1 & - \end{array}\right]$$

Here are the first steps we take

1. Notice the e4 already has 1 at the start, so the first thing we are going to do is to swap e1 and e4 $e_1 \leftrightarrow e_4$
2. Once we have 1 at the top left it is easy to make the rest of the column 0, let's perform two operations to make the the rest of the column 0

$$-3e_1 + e_2 \to e_2$$

$$-2e_1 + e_4 \to e_4$$

$$\begin{bmatrix} 2 & 4 & 2 & 1 & 8 \\ 3 & 1 & 1 & 0 & 4 \\ 0 & 2 & 2 & 0 & 2 \\ 1 & 0 & 0 & 1 & 3 \end{bmatrix} \xrightarrow{e1 \leftrightarrow e4} \begin{bmatrix} 1 & 0 & 0 & 1 & 3 \\ 3 & 1 & 1 & 0 & 4 \\ 0 & 2 & 2 & 0 & 2 \\ 2 & 4 & 2 & 1 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 3 \\ 3 & 1 & 1 & 0 & 4 \\ 0 & 2 & 2 & 0 & 2 \\ 2 & 4 & 2 & 1 & 8 \end{bmatrix} \xrightarrow{-2e_1 + e_4 \to e_4} \begin{bmatrix} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 1 & -3 & -5 \\ 0 & 2 & 2 & 0 & 2 \\ 0 & 4 & 2 & -1 & 2 \end{bmatrix}$$

Once we have zero out the entire column one we now have a smaller problem shown in red.

We repeat the process with a smaller matrix.

Given the smaller matrix we try to get a 1 at the top left of the red portion. Luckily it is already a 1.

1. We now proceeds to 0 out the rest of the column. To do so we perform two more operations and $-4e_2 + e_4 \rightarrow e_4$
2. Remember we are trying to make all the lower triangle values to be 0. since e3 is already in good shape we swap that with the e4 in the second step

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 1 & -3 & -5 \\ 0 & 2 & 2 & 0 & 2 \\ 0 & 4 & 2 & -1 & 2 \end{bmatrix} \xrightarrow[-4e_2 + e_4 \rightarrow e_4]{-2e_2 + e_3 \rightarrow e_3} \begin{bmatrix} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 1 & -3 & -5 \\ 0 & 0 & 0 & 6 & 12 \\ 0 & 0 & -2 & 11 & 22 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 1 & -3 & -5 \\ 0 & 0 & 0 & 6 & 12 \\ 0 & 0 & -2 & 11 & 22 \end{bmatrix} \quad e_3 \leftrightarrow e_4 \quad \begin{bmatrix} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 1 & -3 & -5 \\ 0 & 0 & -2 & 11 & 22 \\ 0 & 0 & 0 & 6 & 12 \end{bmatrix}$$

1. Since we want the diagonal elements to be all 1s we divide e3 by -2 and e4 by 6

$$\xrightarrow[e_4/6 \rightarrow e_4]{e_3/(-2)\rightarrow e_3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 1 & -3 & -5 \\ 0 & 0 & 1 & -5.5 & -11 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right] \longleftarrow$$

We achieved the minimum form to obtain the solution. This form is called the Row Echelon Form

The Row Echelon Form (REF).

We have achieved the REF when

1. The bottom triangle are all 0s
2. The diagonal are all 1s
3. Notice that the upper triangle does not have to have all 0s

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 1 & -3 & -5 \\ 0 & 0 & 1 & -5.5 & -11 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

This is the minimum manipulation that allows us to easily obtain the solutions.

Practice 3

Solve the systems of linear equations using Gaussian elimination law

1)

$$\left[\begin{array}{cccc|c} 3 & 2 & 1 & -1 & 1 \\ 2 & 0 & 4 & -2 & -16 \\ 3 & -3 & -1 & -3 & 8 \\ -4 & 4 & -1 & 3 & 2 \end{array}\right]$$

2)

$$\left[\begin{array}{cccc|c} -4 & 5 & 0 & 2 & -39 \\ 3 & -4 & 0 & 1 & 20 \\ 0 & 4 & 5 & -2 & -9 \\ 3 & 2 & 2 & 3 & -8 \end{array}\right]$$

3)

$$\left[\begin{array}{cccc|c} -4 & 1 & 5 & 5 & -11 \\ -5 & 4 & 2 & 0 & -9 \\ -2 & 0 & -4 & -2 & 0 \\ 4 & -2 & -2 & -3 & 15 \end{array}\right]$$

4)

$$\left[\begin{array}{cccc|c} 0 & 3 & 2 & -4 & 14 \\ -1 & 3 & -1 & -4 & 23 \\ 3 & 0 & 3 & -2 & 5 \\ 4 & 3 & 4 & -1 & 6 \end{array}\right]$$

5)

$$\left[\begin{array}{cccc|c} 0 & 4 & -2 & -1 & -25 \\ -3 & -2 & -3 & -5 & -15 \\ 4 & 5 & -1 & 2 & -14 \\ -4 & -4 & 5 & -3 & 25 \end{array}\right]$$