Linear Algebra

Lecture-05

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Learning Objectives

- Row Echelon Form (REF): Simplifies matrices to triangular form for easier solving of systems.
- Reduced Row Echelon Form (RREF): Provides the most simplified, unique form to directly read solutions.
- Linear System Not Possible: Identifies inconsistent systems (no solution).
- Linear System with Infinite Solutions: Shows dependent systems where many solutions exist.

Learning outcomes

- REF: Learn to transform a matrix into triangular form and simplify solving.
- RREF: Gain skill to get the unique simplified form and directly read solutions.
- No Solution System: Recognize inconsistent systems in matrix form.
- Infinite Solution System: Identify dependent systems and describe solution sets.

Solving a system of linear equations.

$$\begin{aligned} 2x_1 + 4x_2 + 2x3 + x_4 &= 8 \\ 3x_1 + 1x_2 + 1x_3 + 0x_4 &= 4 \\ 0x_1 + 2x_2 + 2x_3 + 0x_4 &= 2 \\ 1x_1 + 0x_2 + 0x_3 + 1x_4 &= 3 \end{aligned}$$

To augmented matrix →

$$\left[\begin{array}{cccc|c} 2 & 4 & 2 & 1 & 8 \\ 3 & 1 & 1 & 0 & 4 \\ 0 & 2 & 2 & 0 & 2 \\ 1 & 0 & 0 & 1 & 3 \end{array}\right]$$

Final goal →

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & - \\ 0 & 1 & 0 & 0 & - \\ 0 & 0 & 1 & 0 & - \\ 0 & 0 & 0 & 1 & - \end{array}\right]$$

Here are the first steps we take

1. Notice the e4 already has 1 at the start, so the first thing we are going to do is to swap e1 and e4 $e_1 \leftrightarrow e_4$
2. Once we have 1 at the top left it is easy to make the rest of the column 0, let's perform two operations to make the the rest of the column 0

$$-3\boldsymbol{e_1} + \boldsymbol{e_2} \rightarrow \boldsymbol{e_2}$$

$$-2\boldsymbol{e_1} + \boldsymbol{e_4} \rightarrow \boldsymbol{e_4}$$

$$\begin{bmatrix} 2 & 4 & 2 & 1 & 8 \\ 3 & 1 & 1 & 0 & 4 \\ 0 & 2 & 2 & 0 & 2 \\ 1 & 0 & 0 & 1 & 3 \end{bmatrix} \xrightarrow{e1 \leftrightarrow e4} \begin{bmatrix} 1 & 0 & 0 & 1 & 3 \\ 3 & 1 & 1 & 0 & 4 \\ 0 & 2 & 2 & 0 & 2 \\ 2 & 4 & 2 & 1 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 3 \\ 3 & 1 & 1 & 0 & 4 \\ 0 & 2 & 2 & 0 & 2 \\ 2 & 4 & 2 & 1 & 8 \end{bmatrix} \xrightarrow{-2e_1 + e_4 \rightarrow e_4} \begin{bmatrix} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 1 & -3 & -5 \\ 0 & 2 & 2 & 0 & 2 \\ 0 & 4 & 2 & -1 & 2 \end{bmatrix}$$

Once we have zero out the entire column one we now have a smaller

problem shown in red.

We repeat the process with a smaller matrix.

Given the smaller matrix we try to get a 1 at the top left of the red portion. Luckily it is already a 1.

1. We now proceeds to 0 out the rest of the column. To do so we perform two more operations and $-4e_2 + e_4 \rightarrow e_4$

2. Remember we are trying to make all the lower triangle values to be 0. since e3 is already in good shape we swap that with the e4 in the second step

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 1 & -3 & -5 \\ 0 & 2 & 2 & 0 & 2 \\ 0 & 4 & 2 & -1 & 2 \end{bmatrix} \xrightarrow[-4e_2 + e_4 \to e_4]{-2e_2 + e_3 \to e_3} \begin{bmatrix} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 1 & -3 & -5 \\ 0 & 0 & 0 & 6 & 12 \\ 0 & 0 & -2 & 11 & 22 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 1 & -3 & -5 \\ 0 & 0 & 0 & 6 & 12 \\ 0 & 0 & -2 & 11 & 22 \end{bmatrix} \quad e_3 \leftrightarrow e_4 \quad \begin{bmatrix} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 1 & -3 & -5 \\ 0 & 0 & -2 & 11 & 22 \\ 0 & 0 & 0 & 6 & 12 \end{bmatrix}$$

1. Since we want the diagonal elements to be all 1s we divide e3 by -2 and e4 by 6

$$\xrightarrow[e_4/6 \rightarrow e_4]{e_3/(-2)\rightarrow e_3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 1 & -3 & -5 \\ 0 & 0 & 1 & -5.5 & -11 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right] \longleftarrow$$

We achieved the minimum form to obtain the solution. This form is called the Row Echelon Form

The Row Echelon Form (REF).

We have achieved the REF when

1. The bottom triangle are all 0s
2. The diagonal are all 1s
3. Notice that the upper triangle does not have to have all 0s

$$\begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{3} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} & -\mathbf{3} & -\mathbf{5} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & -\mathbf{5.5} & -\mathbf{11} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{2} \end{bmatrix}$$

This is the minimum manipulation that allows us to easily obtain the solutions.

If we write the equations back out, we get.

$$\mathbf{1}x_1 + \mathbf{0}x_2 + \mathbf{0}x_3 + \mathbf{1}x_4 = \mathbf{3}$$
$$\mathbf{0}x_1 + \mathbf{1}x_2 + \mathbf{1}x_3 - \mathbf{3}x_4 = -\mathbf{5}$$
$$\mathbf{0}x_1 + \mathbf{0}x_2 + \mathbf{1}x_3 - \mathbf{5.5}x_4 = \mathbf{11}$$
$$\mathbf{0}x_1 + \mathbf{0}x_2 + \mathbf{0}x_3 + \mathbf{1}x_4 = \mathbf{2}$$

1. Notice that x4 is already known by e4
2. If we know x4 it is very easy to get x3 by e3
3. Similarly, if we know x4, x3, we can easily get x2 from e2
4. Finaly since we know x2, x3, x4 we can easily get x1 from e1

The Reduced Row Echelon Form (RREF)

Once we get the REF then it is very easy to obtain the final version, we want the Reduced Row Echelon Form (RREF)

Looking at the previous matrix by using e4 we can easily 0 out the column 4

- $e_1 - e_4 \rightarrow e_1$. $5.5e_4 + e_3 \rightarrow e_3$
- $e_2 + 3e_4 \rightarrow e_2$

- Finaly we can remove the extra 1 in row 2 by $e_2 - e_3 \to e_2$ now we have the RREF

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 1 & -3 & -5 \\ 0 & 0 & 1 & -5.5 & -11 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow[5.5e_4 + e_3 \to e_3]{e_1 - e_4 \to e_1,\, e_2 + 3e_4 \to e_2} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{e_2 - e_3 \to e_2} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

We now know that the final solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}$$

Practice 1

1) Reduce the matrix to **R.R.E.F**

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 1 & 2 & 2 \end{bmatrix}$$

2) Reduce the matrix to **R.R.E.F**

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 6 & 9 \end{bmatrix}$$

3) Reduce the matrix to **R.R.E.F**

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & -1 & 1 \\ 2 & -1 & 1 & 3 \end{bmatrix}$$

Update of Notation

In traditional textbooks, the notation for converting one matrix into another matrix is like $e_1 - e_2$ for example

$$\left[\begin{array}{cc|c} 2 & 1 & 1 \\ 0 & 2 & 1 \end{array}\right] \xrightarrow{e_1 - e_2} \left[\begin{array}{cc|c} 2 & 0 & 0 \\ 0 & 1 & 1 \end{array}\right]$$

- This notation was how I was taught
- However, after our last class, I realized this notation is the reason why I was so confused
- This is why going forward, we are going to follow a different notation (instead of only declaring the operation, we are also going to identify the row it replace)
- So the same operation is going to be

$$\left[\begin{array}{cc|c} 2 & 1 & 1 \\ 0 & 2 & 1 \end{array}\right] \xrightarrow{e_1 - e_2 \to e_1} \left[\begin{array}{cc|c} 2 & 0 & 0 \\ 0 & 1 & 1 \end{array}\right]$$

We learned Gaussian Elimination in the last Lecture

$$\begin{cases} x_1 + 2x_2 = 1 \\ 2x_1 + 4x_2 = 3 \end{cases}$$ ← a linear system may not always have a unique solution or even have a solution at all.

Now, we are going to look at the 3 possible cases.

1. A good unique solution (learned in the last class)

2. No solution

3. Infinite solutions

Take out a piece of paper and try to solve this problem and identify which of the 3 cases is associated with this problem.

A linear system that is not possible(no solution)

$$\begin{cases} x_1 + 2x_2 = 1 \\ 2x_1 + 4x_2 = 3 \end{cases}$$

You could have also seen that by multiplying e1 by 2 you get

$$\begin{cases} 2x_1 + 4x_2 = 2 \\ 2x_1 + 4x_2 = 3 \end{cases}$$

Since $2x_1 + 4x_2 = 2$ can not simultaneously be 2 and 3, it is not possible

$$\left[\begin{matrix} 1 & 2 \\ 2 & 4 \end{matrix} \middle| \begin{matrix} 1 \\ 3 \end{matrix}\right] \xrightarrow{2e_1 - e_2 \to e_2} \left[\begin{matrix} 1 & 2 \\ 0 & 0 \end{matrix} \middle| \begin{matrix} 1 \\ -1 \end{matrix}\right]$$

The second of the augmented matrix states that

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow \begin{cases} x_1 + 2x_2 = 1 \\ 0x_1 + 0x_2 = -1 \end{cases}$$

No solution can satisfy the linear system of equations

and therefor

$0x_1 + 0x_2 = -1$ or 0 = -1 which is not possible

Example of More Equation than Variables

$$\begin{cases} x_1 + 3x_2 = 1 \\ 2x_1 + x_2 = -3 \\ 2x_1 + 2x_2 = -2 \end{cases} \xrightarrow[-2e_1 + e_3 \to e_3]{-2e_1 + e_2 \to e_2} \begin{cases} x_1 + 3x_2 = 1 \\ -5x_2 = -5 \\ -4x_2 = -4 \end{cases}$$

$$\xrightarrow{-{}^4/_5\, e_2 + e_3 \to e_3} \begin{cases} x_1 + 3x_2 = 1 \\ -5x_2 = -5 \\ 0 = 0 \end{cases}$$

- we have an equation where $0 = 0$, this is not a contradiction, so it is okay.
- A contradiction only happen if there is a contradicting logic, for example
  - $5 = 2$
  - two equations that simultaneously equals to different things.

$$\begin{cases} 2x_1 = 2 \\ 2x_1 = 3 \end{cases}$$

Notice the key difference between the 2 cases

look at the 2 cases this is what we just saw, it has a solution

$$\begin{cases} x_1 + 3x_2 = 1 \\ -5x_2 = -5 \\ -4x_2 = -4 \end{cases} \xrightarrow{-\,{}^4\!/_5\, e_2 + e_3 \to e_3} \begin{cases} x_1 + 3x_2 = 1 \\ -5x_2 = -5 \\ 0 = 0 \end{cases}$$

This case does not have a solution

$$\begin{cases} x_1 + 2x_2 = 1 \\ 2x_1 + 4x_2 = 3 \end{cases} \xrightarrow{2e_1 - e_2 \to e_2} \begin{cases} x_1 + 2x_2 = 1 \\ 0x_1 + 0x_2 = -1 \end{cases}$$

Obvious contradiction (← $0x_1 + 0x_2 = -1$)

If there is a contradiction you can not have a solution.

In this example, we have infinite solutions

$$\begin{aligned} x + y + z - w &= 1 \\ y - z + w &= -1 \\ 0 &= 0 \\ 0 &= 0 \end{aligned}$$

Variables in the beginning of the echelon form are called the **Pivot/leading** terms

The variables that are not leading terms are called the **free variables**

After reducing the system of linear equations into the echelon form, if you have more variables than equations, then it means you have

infinite solutions

Here is another example

$$\begin{bmatrix} 0 & x_2 & x_3 & x_4 & x_5 \\ 0 & 0 & 0 & x_4 & x_5 \end{bmatrix}$$

x2 and x4 are leading variables
And x3 and x 5 are free variables

Third case infinite solutions

Given the linear system of equations

1. We first convert it into augmented matrix format
2. We then convert it to RREF

$$\begin{cases} x + y + z - w = 1 \\ y - z + w = -1 \\ 3x + 6z - 6w = 6 \\ -y + z - w = 1 \end{cases} \xrightarrow{\text{Augmented Matrix}} \begin{bmatrix} 1 & 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & -1 \\ 3 & 0 & 6 & -6 & 6 \\ 0 & -1 & 1 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & -1 \\ 3 & 0 & 6 & -6 & 6 \\ 0 & -1 & 1 & -1 & 1 \end{bmatrix} \xrightarrow[-3e_1 + e_3 \to e_3,]{e_2 + e_4 \to e_4,\ 3e_2 + e_3 \to e_3} \begin{bmatrix} 1 & 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} x_1 + x_2 + x_3 - x_4 = 1 \\ x_2 - x_3 + x_4 = -1 \end{cases} \qquad \begin{cases} x_1 = 1 - x_1 - x_3 + x_4 \\ x_2 = -1 + x_3 - x_4 \end{cases}$$

Example

$$\begin{cases} x_1 + 3x_3 = 7 \\ 2x_1 + x_2 = 2 \\ 2x_2 + 4x_1 + x_3 = 4 \end{cases} \xrightarrow{\text{Augmented matrix}} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 7 \\ 2 & 1 & 0 & 2 \\ 2 & 4 & 1 & 4 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 7 \\ 2 & 1 & 0 & 2 \\ 2 & 4 & 1 & 4 \end{array}\right] \xrightarrow[-2e_1 + e_2 \to e_2]{-2e_1 + e_3 \to e_3} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 7 \\ 0 & 1 & -6 & -12 \\ 0 & 4 & -5 & -10 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 7 \\ 0 & 1 & -6 & -12 \\ 0 & 4 & -5 & -10 \end{array}\right] \xrightarrow{-4e_2 + e_3 \to e_3} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 7 \\ 0 & 1 & -6 & -12 \\ 0 & 0 & 19 & 38 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 7 \\ 0 & 1 & -6 & -12 \\ 0 & 0 & 19 & 38 \end{array}\right] \xrightarrow{\frac{e_4}{19}} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 7 \\ 0 & 1 & -6 & -12 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 7 \\ 0 & 1 & -6 & -12 \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow[-3e_3 + e_1 \to e_1]{6e_3 + e_2 \to e_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Example

$$\begin{cases} x_1 + 3x_3 & = 1 \\ 2x_1 + x_2 + x_3 + x_4 & = 3 \\ 2x_1 + 4x_2 + x_3 & = 2 \\ x_3 + x_4 & = 1 \end{cases} \xrightarrow{\text{Augmented matrix}} \begin{bmatrix} 1 & 0 & 3 & 0 & 1 \\ 2 & 1 & 1 & 1 & 3 \\ 2 & 4 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 1 \\ 2 & 1 & 1 & 1 & 3 \\ 2 & 4 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \xrightarrow[-2e_1 + e_2 \to e_2]{-2e_1 + e_3 \to e_3} \begin{bmatrix} 1 & 0 & 3 & 0 & 1 \\ 0 & 1 & -5 & 1 & 1 \\ 0 & 4 & -5 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 1 \\ 0 & 1 & -5 & 1 & 1 \\ 0 & 4 & -5 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \xrightarrow{-4e_2 + e_3 \to e_3} \begin{bmatrix} 1 & 0 & 3 & 0 & 1 \\ 0 & 1 & -5 & 1 & 1 \\ 0 & 0 & 15 & -4 & -4 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 1 \\ 0 & 1 & -5 & 1 & 1 \\ 0 & 0 & 15 & -4 & -4 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \xrightarrow{e_{3 \leftrightarrow} e_4} \begin{bmatrix} 1 & 0 & 3 & 0 & 1 \\ 0 & 1 & -5 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 15 & -4 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 1 \\ 0 & 1 & -5 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 15 & -4 & -4 \end{bmatrix} \xrightarrow{-15e_3 + e_4 \to e_4} \begin{bmatrix} 1 & 0 & 3 & 0 & 1 \\ 0 & 1 & -5 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & -19 & -19 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 1 \\ 0 & 1 & -5 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & -19 & -19 \end{bmatrix} \xrightarrow{e_4/-19} \begin{bmatrix} 1 & 0 & 3 & 0 & 1 \\ 0 & 1 & -5 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 1 \\ 0 & 1 & -5 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow[e_3 - e_4 \to e_3]{e_2 - e_4 \to e_2} \begin{bmatrix} 1 & 0 & 3 & 0 & 1 \\ 0 & 1 & -5 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 1 \\ 0 & 1 & -5 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow[e_2 + 5e_3 \to e_2]{e_1 - 3e_3 \to e_1} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Example

$$\begin{cases} 2x - 2y & = 0 \\ z + 3w & = 2 \\ 3x - 3y & = 0 \\ x - y + 2z + 6w & = 4 \end{cases} \xrightarrow{\text{Augmented matrix}} \begin{bmatrix} 2 & -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 2 \\ 3 & -3 & 0 & 0 & 0 \\ 1 & -1 & 2 & 6 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 2 \\ 3 & -3 & 0 & 0 & 0 \\ 1 & -1 & 2 & 6 & 4 \end{bmatrix} \xrightarrow{e_{1 \leftrightarrow} e_4} \begin{bmatrix} 1 & -1 & 2 & 6 & 4 \\ 0 & 0 & 1 & 3 & 2 \\ 3 & -3 & 0 & 0 & 0 \\ 2 & -2 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 2 & 6 & 4 \\ 0 & 0 & 1 & 3 & 2 \\ 3 & -3 & 0 & 0 & 0 \\ 2 & -2 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[-2e_1 + e_4 \to e_4]{-3e_1 + e_3 \to e_3} \begin{bmatrix} 1 & -1 & 2 & 6 & 4 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & -6 & -18 & -12 \\ 0 & 0 & -4 & -12 & -8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 2 & 6 & 4 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & -6 & -18 & -12 \\ 0 & 0 & -4 & -12 & -8 \end{bmatrix} \xrightarrow[e_{4/-4}]{e_{3/-6}} \begin{bmatrix} 1 & -1 & 2 & 6 & 4 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 3 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 2 & 6 & 4 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 3 & 2 \end{bmatrix} \xrightarrow[e_4 - e_3 \to e_4,\, e_1 - 2e_3 \to e_1]{e_2 - e_3 \to e_2} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} x_1 - x_2 = 0 \\ x_3 + 3x_4 = 2 \end{cases} \Rightarrow \begin{array}{c} x_1 = x_2 \\ x_3 = 2 - 3x_4 \end{array}$$

Practice 2

1) Solve the following system of equations. State whether it has one solution, no solution, or infinitely many solutions.

$$\begin{aligned} x + 2y - z &= 4 \\ 2x + 4y - 2z &= 8 \\ 3x + 6y - 3z &= 12 \end{aligned}$$

2) Solve the following system of equations. State whether it has one solution, no solution, or infinitely many solutions

$$\begin{aligned} x + 2y + 3z &= 0 \\ 2x + 4y + 6z &= 0 \\ -x - 2y - 3z &= 0 \end{aligned}$$

3) Solve the following system of equations. State whether it has one solution, no solution, or infinitely many solutions

$$\begin{aligned} x + y + z &= 4 \\ 2x + 2y + 2z &= 8 \\ x + y + z &= 4 \end{aligned}$$