Linear Algebra

Lecture-06

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Learning Objectives

- Matrix as a function of vectors – Understand matrices as transformations of vectors.
- Multiplication by 0 – Know that multiplying by 0 gives the zero vector/matrix.
- Matrix version of 1 – Identity matrix leaves vectors unchanged.
- Multiplying by a value – Scale vectors (stretch/shrink, flip).
- Rotation of a vector – Use rotation matrices to rotate vectors in space.
- Permutation Matrix – Rearrange vector elements or matrix rows/columns.
- Transform multiple points – Apply one transformation to many vectors at once.
- Dealing with multiple samples – Represent and process many data samples in matrix form.
- Averaging a vector – Compute mean values of vector components

Learning outcomes

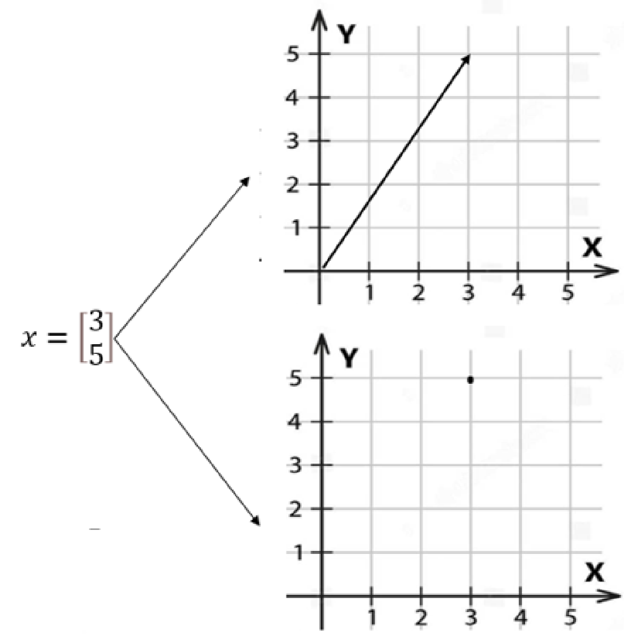
- **Matrix as a function of vectors** – Students will be able to interpret matrices as transformations that act on vectors.
- **Multiplication by 0** – Students will recognize that multiplying any vector or matrix by zero results in the zero vector/matrix.
- **Matrix version of 1 (Identity matrix)** – Students will understand that the identity matrix leaves vectors unchanged.
- **Multiplying by a value (Scalar multiplication)** – Students will learn how scaling a vector changes its magnitude (stretch/shrink) or direction (flip if negative).
- **Rotation of a vector** – Students will apply rotation matrices to rotate vectors in 2D or 3D space.
- **Permutation Matrix** – Students will be able to rearrange vector elements or swap rows/columns of a matrix using permutation matrices

Matrix as a function of vectors

Previously: a vector is a set of numbers enclosed in a bracket

One way to interpret vector is to view them as an arrow with a direction and length(magnitude)

Yet, another interpretation is just a point in space



Practice 1

1) Plot the vector **x=**$\begin{bmatrix}3\\4\end{bmatrix}$ from the origin

2) Plot the vector **y**= $\begin{bmatrix} -4 \\ 4 \end{bmatrix}$ from the origin

3) Plot the vector **w**= $\begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}$ from the origin

4) Plot the vector z= $\begin{bmatrix} 2 \\ -2 \\ 5 \end{bmatrix}$ from the origin

5) Plot the vector w= $\begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}$ from the origin

6) Plot the vector x= $\begin{bmatrix} -1 \\ -2 \\ 5 \end{bmatrix}$ from the origin

7) Plot the vector z= $\begin{bmatrix} -1 \\ 5 \\ 5 \end{bmatrix}$ from the origin

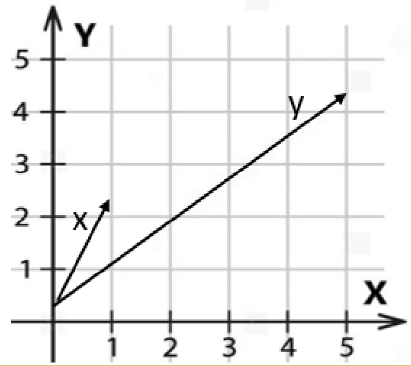
8) Plot the vector y= $\begin{bmatrix} 5 \\ 5 \\ 3 \end{bmatrix}$ from the origin

If you interpret a vector as an arrow what is happening if we multiply a vector by a matrix?

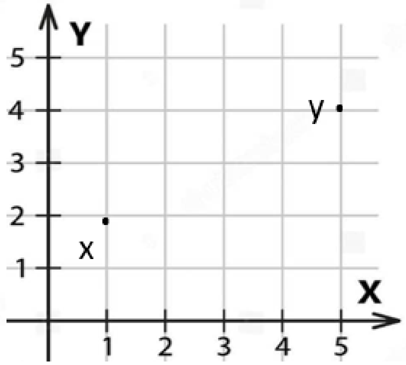
$$x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$Ax = y$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$



It transform to another vector



It move to another location

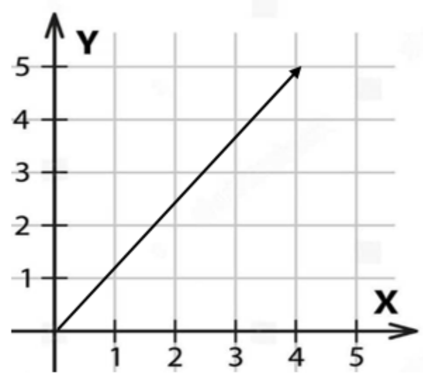
In both cases, the matrix A acts as a function it takes vector x as input and output vector y as output

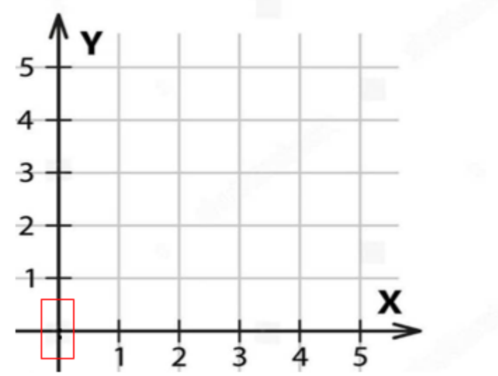
Today's lecture is all about the different ways we can transform a vector into another vector!!!

Multiplication by 0

$$Ax = y$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

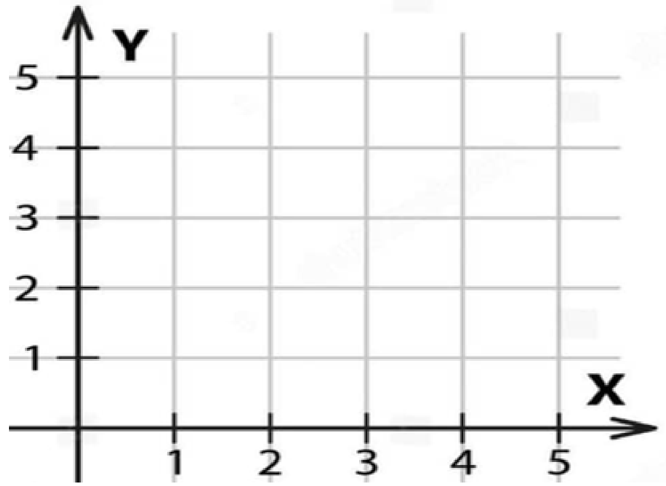




A vector shrink to size 0

The symbol for a 0 matrix is **0** where **0**x = **0**

It is the analogy of a number multiply be 0 It always results to 0



Just like multiplying any number by 0 gives 0, multiplying any vector by a ***zero matrix*** gives the ***zero vector***

When a vector is multiplied by a ***zero matrix***:

- It ***loses all direction and length***
- It becomes the ***zero vector***: a point at the origin (0, 0)
- In visual terms, it's like the vector ***shrinks or collapses*** into nothingness

Matrix version of multiplying 1

$$Ax = y \tag{1}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{2}$$

A matrix that output the same vector as the input is called The identity Matrix

Identity matrix is always 0's everywhere and 1's on the diagonal entries

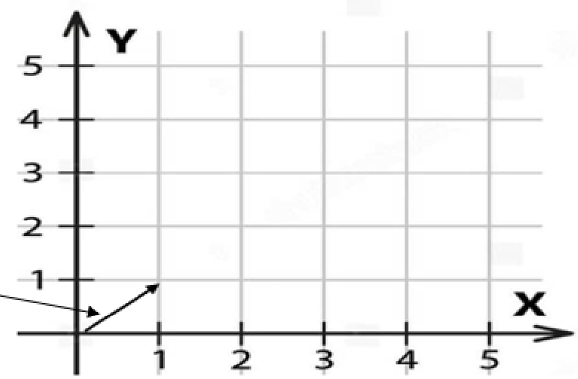
An identity matrix is always a square matrix

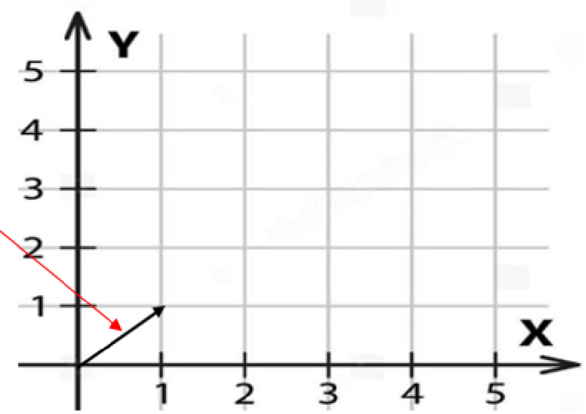
The symbol for identity matrix is

$$I \ where \ Ix = x \tag{3}$$

$$Ax = y \quad (1)$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (2)$$





Nothing change with the vector

Multiplying by a value

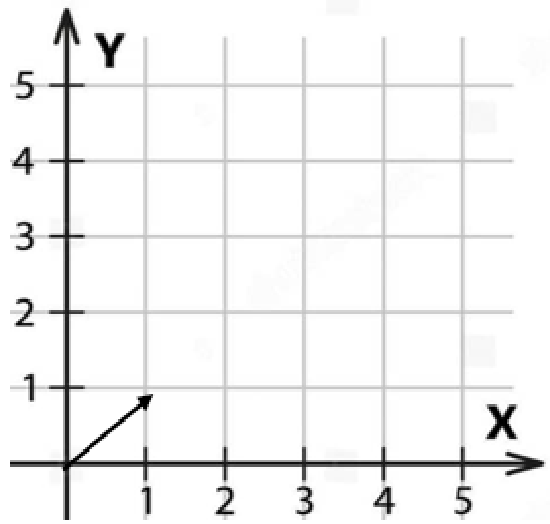
$$Ax = y$$

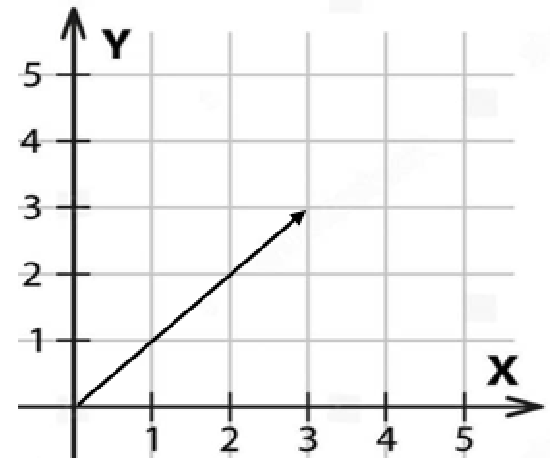
$$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

A matrix that stretches a vector without any rotation is called scalar matrix, but instead of 1's in the diagonal it has a different values in the diagonal

The symbol for scalar matrix is just the number itself

$$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$





It is the analogy of a number multiplying by a constant value

It always result in itself but different.

Rotation of vector

$$Ax = y$$

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = y$$

$$\begin{bmatrix} \cos 45 & \sin 45 \\ -\sin 45 & \cos 45 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix}$$

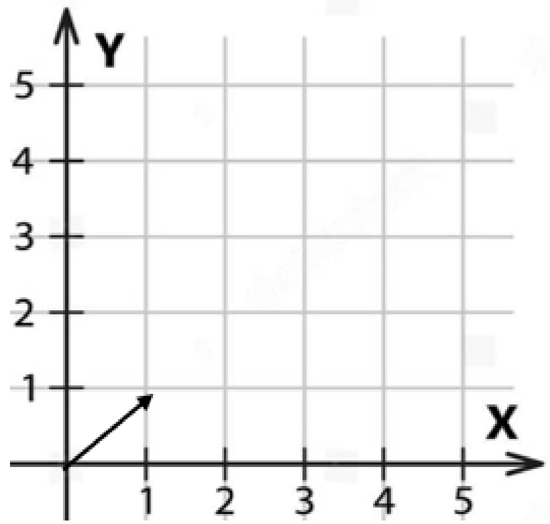
We set that as the angle of the rotation

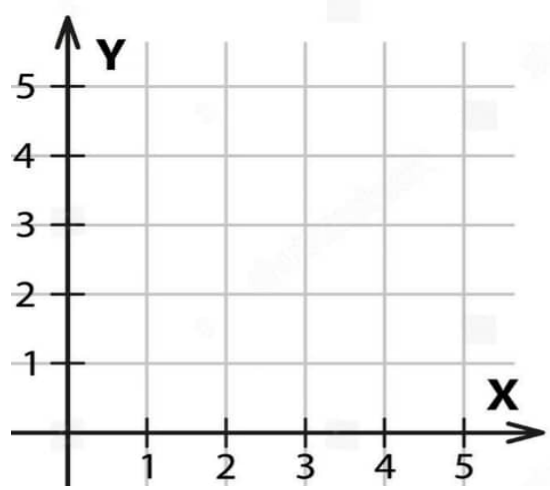
the symbol for rotation matrix is

$$R_\theta \quad \text{where} \quad R_{\pi/4} = y$$

y is x rotated by 45 degree.

The vector $\begin{bmatrix}1\\1\end{bmatrix}$ is rotated by 45 degree





Permutation Matrix

The permutation matrix rearrange the order of elements of input vector.

All 6 Possible 3×3 Permutation Matrices:

1. **Identity permutation (no change)**

$$P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2. **Swap row 1 and row 2**

$$P_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3. Swap row 1 and row 3

$$P_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

4. Swap row 2 and row 3

$$P_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

5. Cycle down: (1→2, 2→3, 3→1)

$$P_5 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

6. Cycle up: (1→3, 3→2, 2→1)

$$P_6 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

The permutation matrix rearrange the order of elements of input vector.

$$\mathrm{Ax} = \mathrm{y}$$

a) $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 4 \end{bmatrix}$

$b)$ $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 3 \\ 2 \end{bmatrix}$

We basically put a one in the location where we want the item to move the item to.