Linear Algebra

Lecture-07

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Learning Objectives

- Apply matrix operations to transform many points at once.
- Analyze and manipulate datasets with multiple samples efficiently.
- Calculate the average/mean vector of given data points.
- Differentiate between translating a single point and an entire object.
- Perform composite transformations by combining matrices.
- Use a centering matrix to re-center data at the origin.
- Implement scaling to change the size of objects.
- Determine the centroid of a set of points or object.

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Transform multiple points simultaneously

Rotating Vectors counterclockwise

To rotate a vector $v = \begin{bmatrix} x \\ y \end{bmatrix}$ by angle θ counterclockwise:

We use this formula

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$u = R(\theta)v = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix}$$

The Clockwise Rotation Matrix

To rotate clockwise, you simply use a negative angle (-θ). Rotating by -θ is mathematically identical to rotating counterclockwise by θ

This gives us the clockwise rotation matrix:

$$R(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

To rotate a vector $v = \begin{bmatrix} x \\ y \end{bmatrix}$ by angle θ clockwise

$$u = R(\theta)v = \begin{bmatrix} x\cos\theta + y\sin\theta \\ -x\sin\theta + y\cos\theta \end{bmatrix}$$

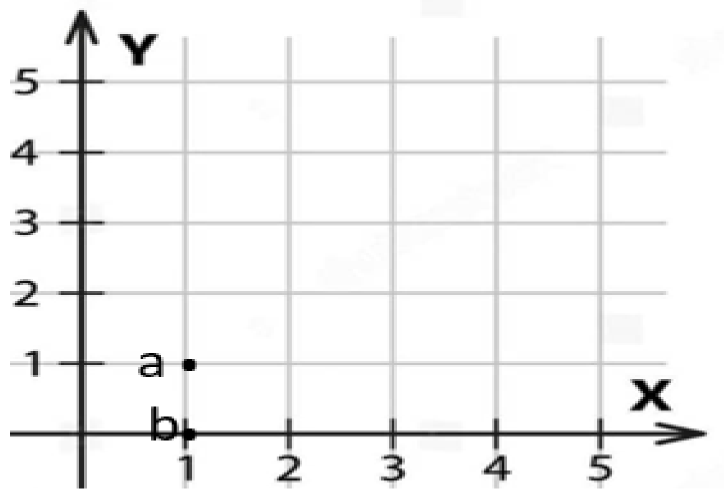
Example

When you have multiple points, you want to simultaneously transform them, you can stack them as columns in a right matrix. For example, given 2 points

$x_1 = [1 \quad 1]^T$ and $x_2 = [1 \quad 0]^T$, we can simultaneously rotate both points by stacking them into a right matrix

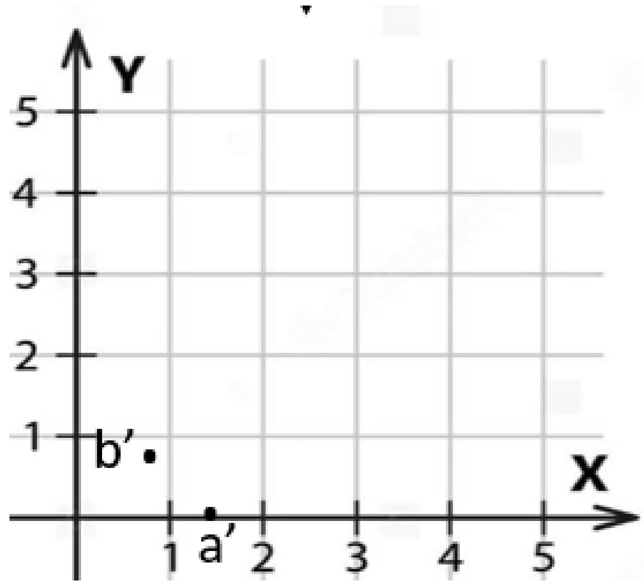
$$AX = Y \quad (1)$$

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = Y \quad (2)$$



$$\begin{bmatrix} \cos 45 & \sin 45 \\ -\sin 45 & \cos 45 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} \end{bmatrix} \qquad (3)$$

Notice how both points are rotated together this imply that we can have any number of points by stacking them together we cam perform one single matrix multiplication and transform all points. And this matrix multiplication is standard way of transforming multiple points.

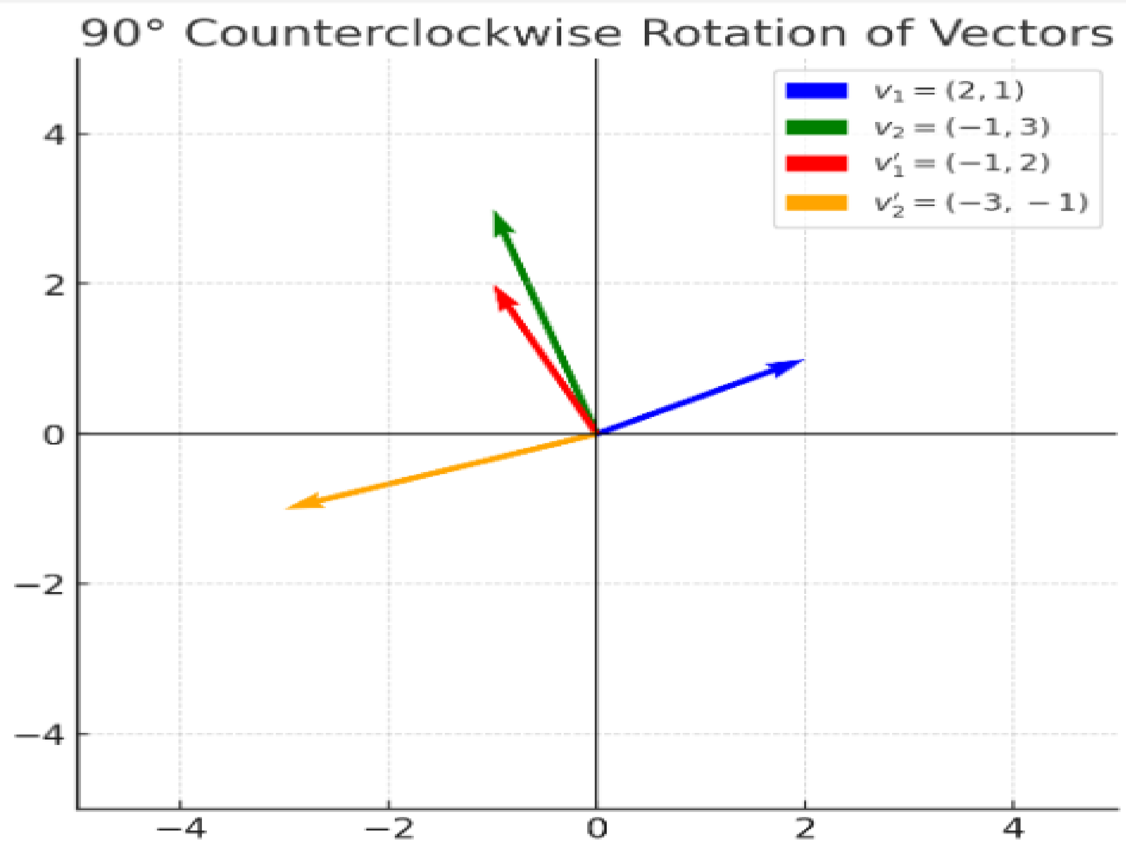


Y
5
4
3
2
1
b'
X
a'
1
2
3
4
5

Example

Let $v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and$v_1 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ rotate both vectors by 90° counterclockwise**.**

$$\begin{bmatrix} \cos 90 & -\sin 90 \\ \sin 90 & \cos 90 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} -1 & -3 \\ 2 & -1 \end{bmatrix}$$



Example

Rotate $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$ by 90 degree

Solution

$$\begin{bmatrix}\cos 90 & -sin90\\ sin90 & \cos 90\end{bmatrix}$$

$$\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$$

Rotation of 3D vector

1) Rotation about the **x-axis** (by angle θ):

$$R_x(\theta) = \begin{bmatrix}1 & 0 & 0\\ 0 & cos\theta & -sin\theta\\ 0 & sin\theta & cos\theta\end{bmatrix}$$

It will Rotates the **y-z plane**, keeps the x-coordinate unchanged

2) Rotation about the **y-axis** (by angle θ):

$$R_y(\theta) = \begin{bmatrix} cos\theta & 0 & sin\theta \\ 0 & 1 & 0 \\ -sin\theta & 0 & cos\theta \end{bmatrix}$$

It will rotates the **x-z plane**, keeps the y-coordinate unchanged.

3) Rotation about the z-axis (by angle θ)

$$R_z(\theta) = \begin{bmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

It will rotates the x-y plane, keeps the z-coordinate unchanged.

Example

We have two vectors:

$$v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad v_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

We want to **scale them by 2 in the x-direction** and **by 3 in the y-direction**.

Put them in a matrix

$$V = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$$

(Each column is one vector.)

Write the transformation matrix

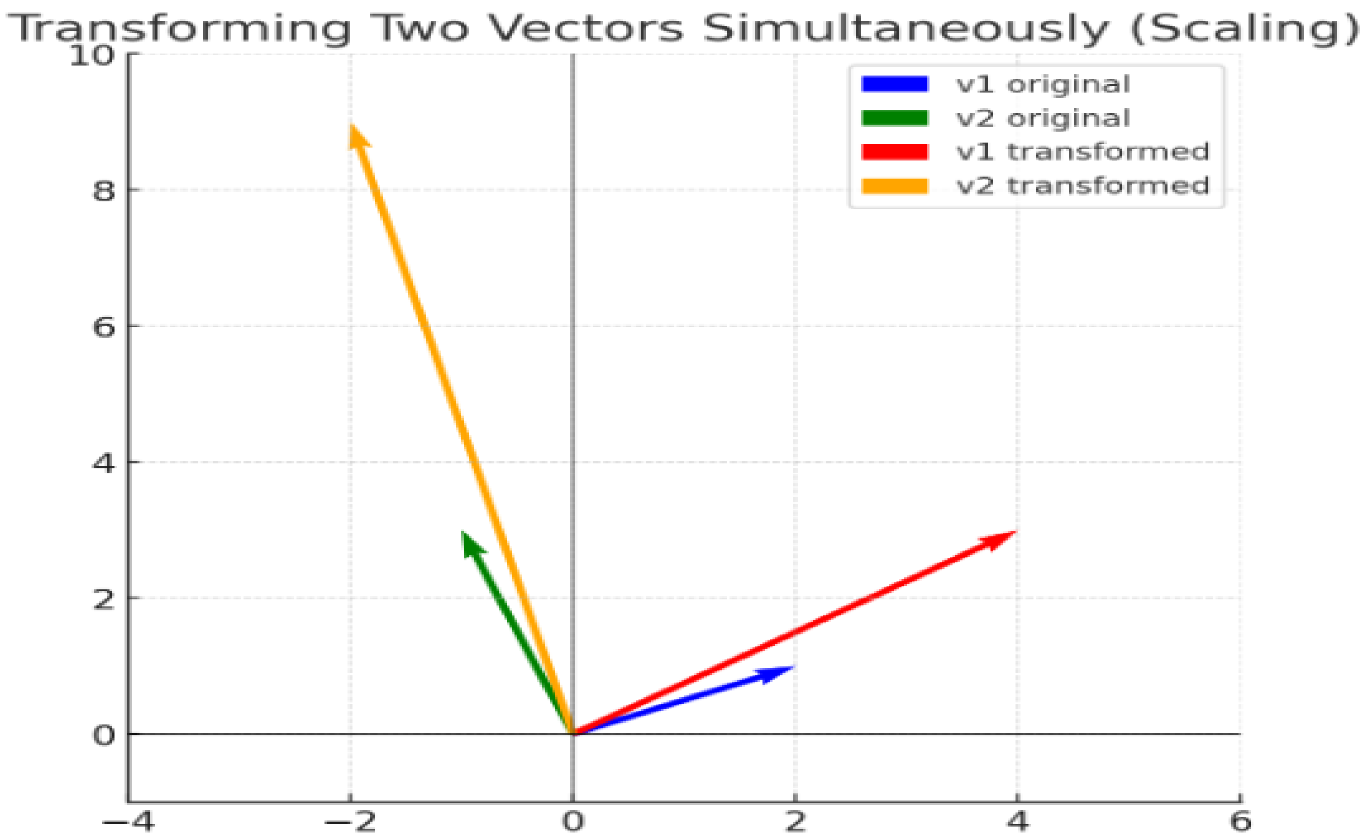
Scaling:

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

Multiply $AV = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 3 & 9 \end{bmatrix}$

First transformed vector = $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$ (scaled version of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$)

Second transformed vector = $\begin{bmatrix} -2 \\ 9 \end{bmatrix}$ (scaled version of $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$)



Blue & Green = original vectors $\begin{bmatrix}2\\1\end{bmatrix}$ and $\begin{bmatrix}-1\\3\end{bmatrix}$.

Red & Orange = transformed vectors $\begin{bmatrix}4\\3\end{bmatrix}$ and $\begin{bmatrix}-2\\9\end{bmatrix}$ after scaling

Dealing with multiple samples

Given $X \in R^{7\times1000}$

The matrix represent the running record if 1000 peoples in this week(by miles)

$$X = \begin{bmatrix} 2 & 1 & 0 & \dots & 3 \\ 3 & 4 & 3 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 4 & \dots & 6 \end{bmatrix}$$

If Y represent the total miles ran in this week and $AX = Y,$ then what is A?Notice if you write $R^{7\times1000}$ matrix it would be annoying The matrix notation allows us to conceptualize and manipulate large data easily

You can accomplish this with A as a matrix of all 1s.

$$A \in R^{1\times7} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

A matrix of 1s is used very often for

- Summation
- Average
- Subtraction
- Centering

It has its own special symbols

$$\mathbf{1} \in R^{1\times 7}$$

Averaging a vector

We previously saw the summation vector **1**. it basically sum up all values from a vector.

$$\mathbf{1}_3^T x = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 2$$

If you have multiple vectors, it can also sum up each individual element simultaneously

$$\mathbf{1}_3^T \mathrm{X}^{\mathrm{T}} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 1 & 0 & 1 \end{bmatrix} = [2 \;\; 5 \;\; 1]$$

We can extend this idea to find the average of a vector

$$\frac{\mathbf{1}}{\mathbf{3}}\mathbf{1}_3^T x = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \frac{2}{3}$$

Or

$$\frac{\mathbf{1}}{\mathbf{3}}\mathbf{1}_3^T \mathrm{X}^{\mathrm{T}} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{5}{3} & \frac{1}{3} \end{bmatrix}$$

Practice 1

1. If $x = \begin{bmatrix} 2 & 5 & 3 \\ 1 & 3 & 2 \\ 3 & 6 & 8 \\ -2 & -2 & 3 \end{bmatrix}$ find the average of x?

2. If $y = \begin{bmatrix} 1 & 6 & 3 \\ 3 & 4 & 5 \end{bmatrix}$ find the average of y?

3. If z=$\begin{bmatrix} 12 & 2 \\ 3 & 3 \\ 0 & 4 \end{bmatrix}$find the average of z?

Moving a point vs object

If you have a point, and you would like to move that somewhere you can add that vector to another vector.

Example given

$$x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Then we can move it to point $y = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ by

$$y = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

If you want to move multiple points simultaneously by $z = \begin{bmatrix} 2 & 1 \end{bmatrix}^T$ it has to be done slightly differently. Given 7 points

$$x_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, x_2 = \begin{bmatrix} 4 \\ 0 \end{bmatrix}, x_3 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, x_4 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, x_5 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, x_6 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, x_7 = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

We would first stack theme together

$$X^T = \begin{bmatrix} 2 & 4 & 3 & 3 & 2 & 4 & 3 \\ 0 & 0 & 1 & 2 & 2 & 2 & 3 \end{bmatrix}$$

Then we perform the addition to $z\mathbf{1}_7^T$

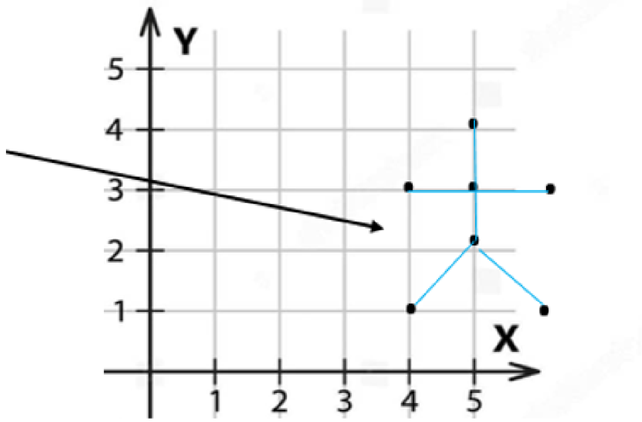
$$Y^T = z\mathbf{1}_7^T + X^T$$

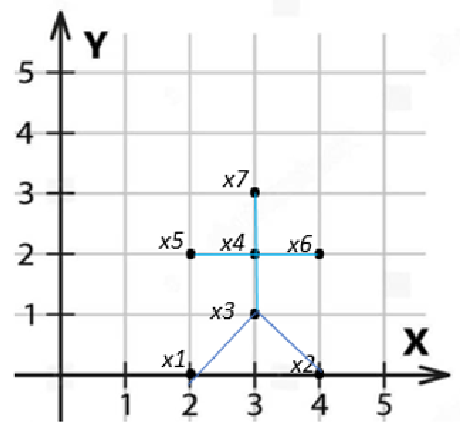
$$= \begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 4 & 3 & 3 & 2 & 4 & 3 \\ 0 & 0 & 1 & 2 & 2 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 4 & 3 & 3 & 2 & 4 & 3 \\ 0 & 0 & 1 & 2 & 2 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 6 & 5 & 5 & 4 & 0 & 5 \\ 1 & 1 & 2 & 3 & 3 & 3 & 4 \end{bmatrix}$$

The result of the above transformation is shown in this graph





Combination of transformation

Remember when we rotated a vector

$$\begin{bmatrix} \cos 45 & \sin 45 \\ -\sin 45 & \cos 45 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix}$$

We can combine operations together, here we first rotate a vector then and then scale it by 2

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \cos 45 & \sin 45 \\ -\sin 45 & \cos 45 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix}$$

This is equivalent to multiplying the 2 matrices together first

$$\begin{bmatrix} 2\cos 45 & 2\sin 45 \\ -2\sin 45 & 2\cos 45 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2\sqrt{2} \\ 0 \end{bmatrix}$$

Here we multiply the 2 vectors by 2 then we permuted the order

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

This is equivalent to multiplying the 2 matrices first

$$\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$