Linear Algebra

Lecture 09

Vector spaces, Row spaces, Column spaces.

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Objectives

Vector Spaces

- Understand the concept of vector spaces and their properties.
- Learn vector addition and scalar multiplication.
- Identify subspaces, span, basis, and dimension.

Row Spaces

- Understand the row space of a matrix.
- Analyze relationships among matrix rows.
- Use row operations to determine rank and linear dependence.

Column Spaces

- Understand the column space of a matrix.
- Analyze how columns generate vectors through linear combinations.
- Determine whether a system of equations has a solution.

Outcomes

Vector Spaces

- Apply vector space properties to solve problems.
- Determine basis, dimension, and span of a vector space.
- Test vectors for linear independence.

Row Spaces

- Find the row space of a matrix.
- Determine the rank of a matrix.
- Identify dependent and independent equations.

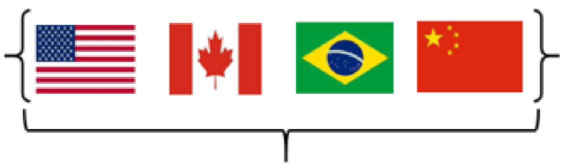
Column Spaces

- Find the column space of a matrix.
- Determine whether a vector belongs to the column space.
- Analyze the solvability of systems of linear equations.

Set: A set is a collection of distinct elements.

Examples

$\{\text{dog}, \ \text{cat}, \ fish, \ bird\}$ Set of animals



A set of flags

$\{1,2,3,4,5 \ldots\}$ Set of numbers

If nothing is inside the set it is an empty set

$$\emptyset = \{\ \}$$

A set is denoted by a cursive capital letter $\quad \mathrm{X} = \{\mathrm{a}, \mathrm{b}, \mathrm{c}\}$

$$\chi, y, S$$

A space is just a set with some defined properties

Just like vector sets by itself does not mean anything and we can not do much with it. For example, how do you compute {  +  }

We do not know because it was not previously defined

This is where the concept of space come in

Space is a set where certain structures are defined

For example {  +  }

This could mean the summation of the population of the 2 country

vector space

vector space is just a collection of objects (which we call vectors) where you can do two main things without breaking the rules of the space:

1. **Add** any two vectors together to get a new vector that is still in the space.
2. **Scale** any vector by multiplying it by a number (a scalar) to get a new vector that is still in the space.

The 8 Rules of the vector space

If a set of objects can't pass all eight of these tests under addition and scalar multiplication, it is not a vector space.

Vector Addition Rules

These rules ensure that adding vectors together behaves predictably.

- **Commutativity:** The order doesn't matter.

If u, v are vectors

$$u + v = v + u$$

- **Associativity:** Grouping doesn't matter when adding three or more.

$$(u + v) + w = u + (v + w)$$

- **Identity Element:** There is a unique "zero vector" 0 that leaves any vector unchanged.

$$u + 0 = u$$

- **Inverse Element:** Every vector has an exact opposite that pulls it back to zero.

$$u + (-u) = 0$$

Scalar Multiplication Rules

- **Distributivity over Scalar Addition:** Scaling a vector by two added numbers is the same as scaling it by each separately and adding the results.

$$(c + d)u = cu + du$$

- **Compatibility of Multiplication:** Scaling a vector consecutively works exactly like multiplying the scalars together first.

$$c(du) = (cd)u$$

- **Identity Scalar:** Multiplying any vector by the number 1 leaves it completely unchanged.

$$1u = u$$

EXAMPLE

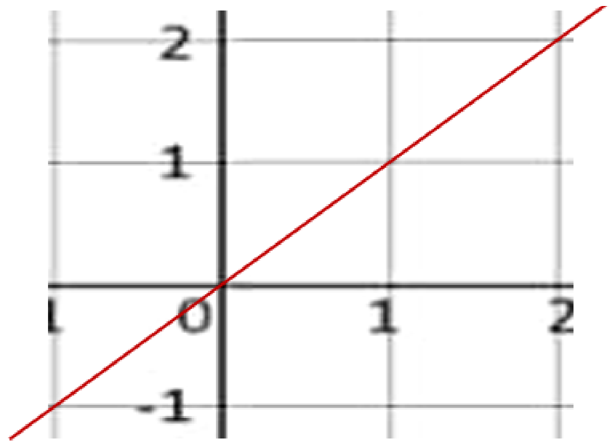
The 2D Plane ($\mathbf{R}^2$): Think of standard x, y coordinates. If you take the arrow $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and add it to $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$, you get $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$, which is still on the flat plane. If you multiply $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ by 3, you $\begin{bmatrix} 3 \\ 6 \end{bmatrix}$ still on the plane.

A set can be defined by a vector

Given the vector $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ we can define a space as all possible points reachable by some constant multiplying the vector.

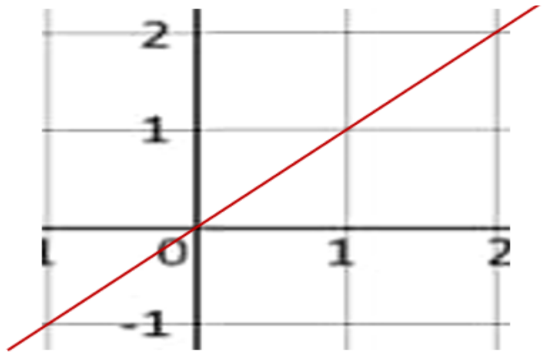
$$v = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

for example, $v = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad or \quad$ $v = -2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$



Since this set is defined by a vector, so it come with all properties of vector addition subtraction, hence it is a space with defined operations.

You can imagine every point on this line as part of the set.



This set has infinite points on it

This is an example of 2-d space

linear combination of the vectors

DEFINITION

If w is a vector in R^n, then w is said to be a linear combination of the vectors v_1, v_2 ,..... v_r in R^n if it can be expressed in the form

w = k_1 v_1 + k_2 v_2 +...+ k_r v_r where k_1, k_2 ,..., k_r are scalars.

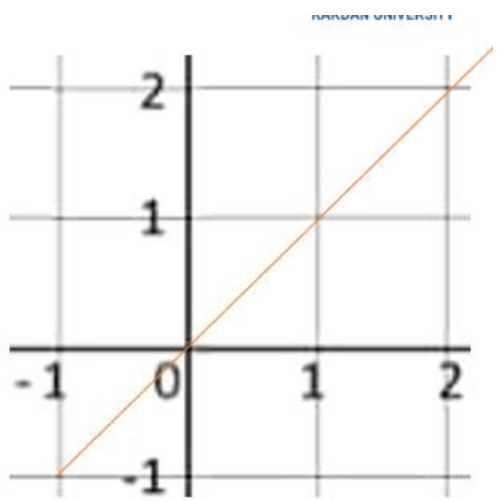
These scalars are called the coefficients of the linear combination.

Span

This space is the span of vector $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Span is very important concept

- This is the set of all possible points reachable by one or multiple scaled vectors



the span of $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

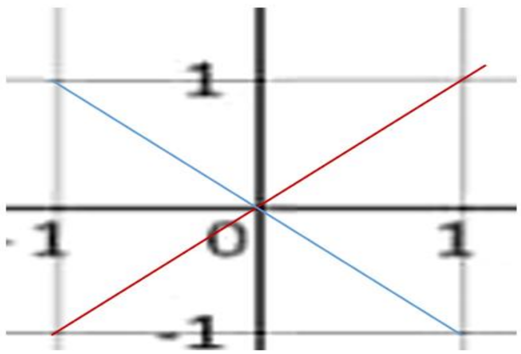
$$\begin{bmatrix}0.5\\0.5\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1.5\\1.5\end{bmatrix}, \begin{bmatrix}2\\2\end{bmatrix}, \begin{bmatrix}5\\5\end{bmatrix}, \begin{bmatrix}100\\100\end{bmatrix} \dots \dots, \begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}-0.5\\-0.5\end{bmatrix}, \begin{bmatrix}-1\\-1\end{bmatrix}, \begin{bmatrix}-2\\-2\end{bmatrix}\dots\dots\dots\dots$$

All these vectors are the same because they are all along the same line.

The red line is the span of $x = \begin{bmatrix}1\\1\end{bmatrix}$

The blue line is the span of $\mathrm{x} = \begin{bmatrix}-1\\1\end{bmatrix}$

they both intersect at point (0,0)



Span is a very important concept

it is the set all point reachable by one or multiple scaled vectors

The span of multiple vectors

With a vector v its span is

$\boldsymbol{S} = \text{span}(v) = \{\alpha v : \alpha \in \mathbb{R}\}$ This notation is called the set builder notation.

It defines a set such that α can be any real number.

You may sometime see a line instead.

But you can also have multiple vectors

$v_1, v_2, v_3, \dots,$ its span would be

$$S = \text{span}(v_1, v_2, v_3 \dots,) = \{(av_1, bv_2 \dots): a, b \in \mathbb{R}\}$$

For example, you have 2 vectors

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

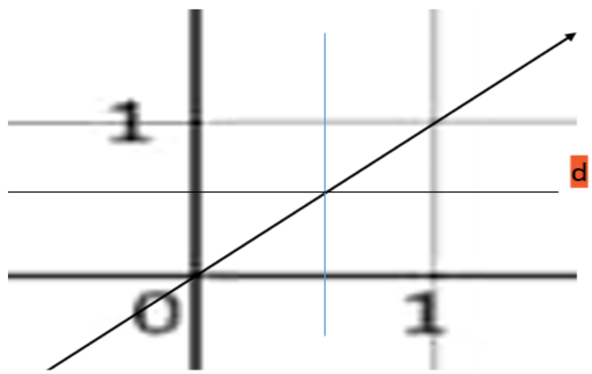
Their span would cover the entire 2D space

For example, the point d is in the span because

$$d = v_1 + {}^{1}/_{2}\, v_2$$

The vectors used to create the span of a space are called basis

$$\begin{bmatrix} 1.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$



Every single point on this 2D plane can be achieved by some combination of the 2 vectors

How do you know if a point is in the span for a set of vectors?

By definition if a set of vectors can scale and sum up to a target point, then the target is in the span

Given $\{v_1, v_2, v_3\}$ and the target vector u

$$u \in \text{span}\{v_1, v_2, v_3\}\,?$$

This is equivalent to asking if there exists

a, b, c such that

$$av_1 + bv_2 + cv_2 = u$$

For example

$$a\begin{bmatrix}1\\0\end{bmatrix} + b\begin{bmatrix}1\\1\end{bmatrix} + c\begin{bmatrix}2\\2\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix}$$

This gets harder if we have a lot of vectors

Matrix multiplication is the same operation

$$a\begin{bmatrix}1\\0\end{bmatrix} + b\begin{bmatrix}1\\1\end{bmatrix} + c\begin{bmatrix}2\\2\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix}$$

$$\begin{cases} a + b + 2c = 2 \\ \quad b + 2c = 1 \end{cases}$$

$$\longrightarrow$$

$$\begin{bmatrix}1 & 1 & 2\\0 & 1 & 2\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix}$$

$$\begin{cases} a + b + 2c = 2 \\ \quad b + 2c = 1 \end{cases}$$

Identifying if a point is in the span of a set of vectors is identical to solving a system of linear equations.

If a solution exist (infinite or unique), then the target is in the span

This is why gaussian elimination is so important

Matric multiplication is the same operation

$$a\begin{bmatrix}1\\0\end{bmatrix}+b\begin{bmatrix}1\\1\end{bmatrix}+c\begin{bmatrix}2\\2\end{bmatrix}=\begin{bmatrix}2\\1\end{bmatrix}$$

$$\begin{bmatrix}1 & 1 & 2\\0 & 1 & 2\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}2\\1\end{bmatrix}$$

$$\begin{cases}a+b+2c=2\\ \quad b+2c=1\end{cases}$$

$$\begin{cases}a+b+2c=2\\ \quad b+2c=1\end{cases}$$

Here, we are interpreting each column as a vector

The space span by these columns vectors is called column space

subspace

A **subspace** is a concept from linear algebra. It is a subset of a vector space that is itself a vector space under the same operations of vector addition and scalar multiplication.

For a set W to be a subspace of a vector space V, it must satisfy three conditions:

- **Contains the zero vector**

The vector $\mathbf{0}$ must be in W.

- **Closed under addition**

If $\mathbf{u}$ and $\mathbf{v}$ are in W, then $\mathbf{u} + \mathbf{v}$ must also be in W.

- **Closed under scalar multiplication**

If $\mathbf{u}$ is in W and c is any scalar, then $c\mathbf{u}$ must also be in W.

Subspaces of $\mathbb{R}^1$

$\mathbb{R}^1$ is just the set of real numbers.

There are only two subspaces:

- The zero subspace:

$$\{0\}$$

- The whole space:

$$\mathbb{R}$$

Subspaces of $\mathbb{R}^2$

A subspace of $\mathbb{R}^2$ can have dimension 0, 1, or 2.

1. Dimension 0: The zero subspace

$$\{(0, 0)\}$$

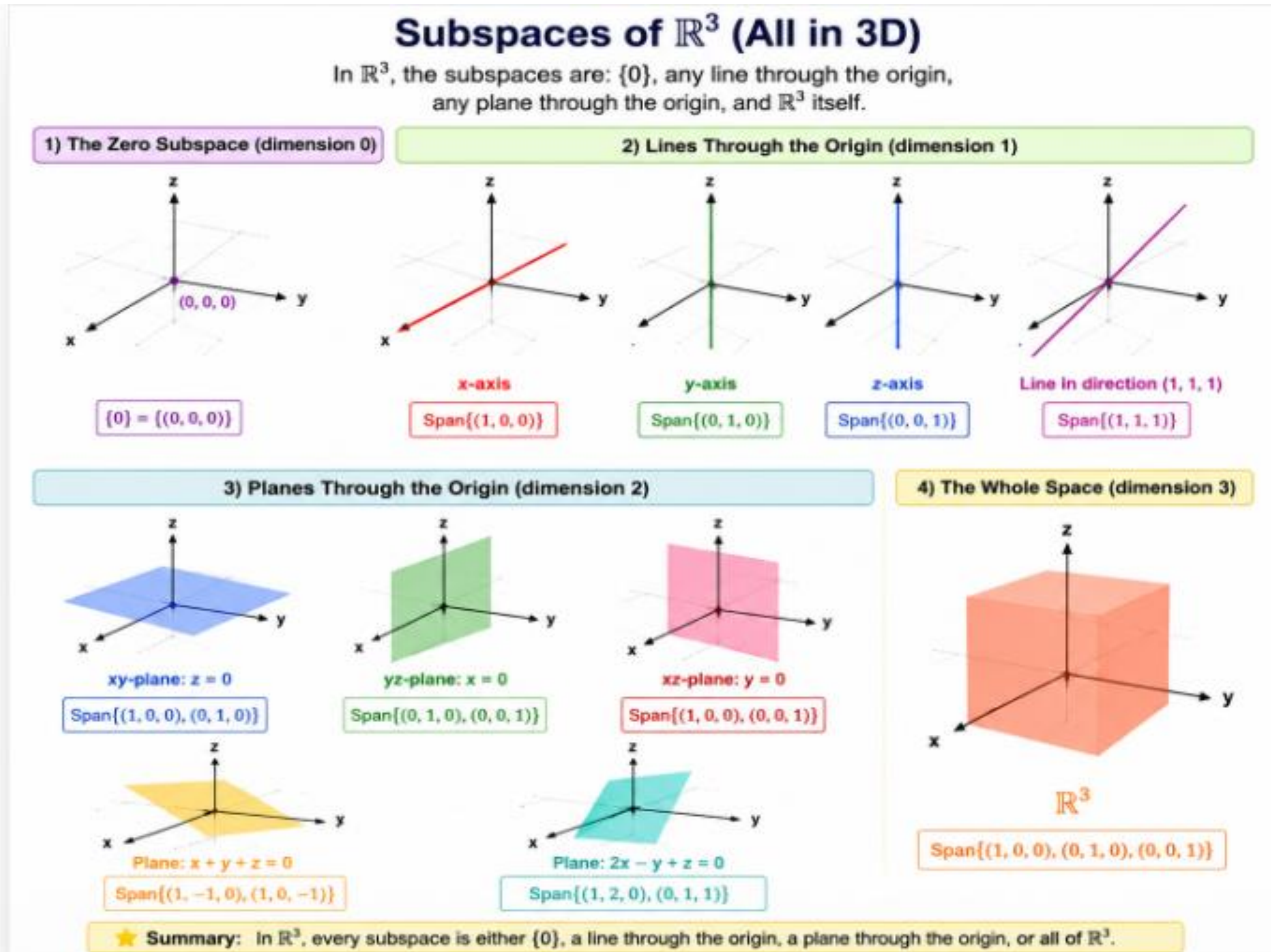
2. Dimension 1: Any line through the origin

3. Dimension 2: The whole space

$$\mathbb{R}^2$$

Common subspaces of $\mathbb{R}^3$

- The zero subspace: $\{(0, 0, 0)\}$
- Any line through the origin
- Any plane through the origin
- The entire space $\mathbb{R}^3$



Subspaces of $\mathbb{R}^3$ (All in 3D)
In $\mathbb{R}^3$, the subspaces are: {0}, any line through the origin,
any plane through the origin, and $\mathbb{R}^3$ itself.
1) The Zero Subspace (dimension 0)
(0, 0, 0)
{0} = {(0, 0, 0)}
2) Lines Through the Origin (dimension 1)
x-axis
Span{(1, 0, 0)}
y-axis
Span{(0, 1, 0)}
z-axis
Span{(0, 0, 1)}
Line in direction (1, 1, 1)
Span{(1, 1, 1)}
3) Planes Through the Origin (dimension 2)
xy-plane: z = 0
Span{(1, 0, 0), (0, 1, 0)}
yz-plane: x = 0
Span{(0, 1, 0), (0, 0, 1)}
xz-plane: y = 0
Span{(1, 0, 0), (0, 0, 1)}
Plane: x + y + z = 0
Span{(1, −1, 0), (1, 0, −1)}
Plane: 2x − y + z = 0
Span{(1, 2, 0), (0, 1, 1)}
4) The Whole Space (dimension 3)
$\mathbb{R}^3$
Span{(1, 0, 0), (0, 1, 0), (0, 0, 1)}
Summary: In $\mathbb{R}^3$, every subspace is either {0}, a line through the origin, a plane through the origin, or all of $\mathbb{R}^3$.

Column Space

Definition

Suppose

$$A = [a_1 \quad a_2 \quad \cdots \quad a_n]$$

where $a_1, a_2, \dots, a_n$ are the columns of A.

The column space of A, denoted by Col(A) ,is the set of all linear combinations of the columns of A.

$$\text{Col}(A) = \mathit{span}\{a_1, a_2, \dots, a_n\}$$

Means

Col$(A) = c_1 a_1 + c_2 a_2 + \cdots + c_n a_n : c_1 c_2 \dots c_n \in \mathrm{R}$

So the column space is literally the span of the columns.

Or

Definition The column space of a matrix $A$is the set of all possible linear combinations of its independent column vectors. If the columns of $A$are $a_1, a_2, \dots, a_n$,then

$$\mathrm{Col}(A) = span\{a_1, a_2, \dots, a_n\} = \{c_1a_1 + c_2a_2 + \cdots + c_na_n : c_i \in \mathbb{R}\}.$$

Thus, the column space consists of all vectors that can be generated from the columns of Ausing linear combinations.

Geometric Interpretation

1)One Independent Column

$$\mathrm{Col}(A) = span\{v\}$$

Result: a **line**.

2)Two Independent Columns

$$\mathrm{Col}(A) = span\{v_1, v_2\}$$

Result: a **plane** through the origin.

3)Three Independent Columns in $\mathbb{R}^3$

$$\text{Col}(A) = span\{v_1, v_2, v_3\}$$

Result: entire $\mathbb{R}^3$.

Example

Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$$

Columns are

$$a_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \; a_2 = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

Notice

$$a_2 = 2a_1$$

Therefore the columns are dependent.

Hence

$$\text{Col}(A) = span\left\{\begin{bmatrix}1\\3\end{bmatrix}\right\}$$

or $c_1\begin{bmatrix}1\\3\end{bmatrix} + c_2\begin{bmatrix}2\\6\end{bmatrix}$ where c_1, c_2 from R

Every vector in the column space looks like

$$c\begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix}c\\3c\end{bmatrix}$$

So the column space is a **line through the origin**.

Example

$$A = \begin{bmatrix}1 & 2\\2 & 4\end{bmatrix}$$

Step 1: Row reduce

$$\begin{bmatrix}1 & 2\\2 & 4\end{bmatrix} \to \begin{bmatrix}1 & 2\\0 & 0\end{bmatrix}$$

Pivot is in Column 1.

Step 2: Go back to ORIGINAL matrix

Column 1 of the original matrix is

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Basis for Col(A):

$$Col(A) = \left\{ c \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$

Dimension = 1

So its 1-D Subspace of 2-D

Example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 1 \end{bmatrix}$$

Step 1: Row reduce

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 1 \end{bmatrix}$$

R2−2R1 $\rightarrow$ R2
$R_3 - R_1 \rightarrow$ R3

gives

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & -1 & -2 \end{bmatrix}$$

Swap rows

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

Pivot columns are 1 and 2.

Step 2: Take columns 1 and 2 from ORIGINAL matrix

Column 1 and 2 :

$$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} \text{ Basis: span } \left(\left\{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}\right\}\right)$$

Dimension = 2

So it is a 2-D subspace of 3-D

Example 3 (Plane)

Consider

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

Columns are

$$a_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},\ a_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Column space:

$$\text{Col}(A) = span \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\} \text{or} \left\{ c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Any vector in the column space is

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_1 + c_2 \end{bmatrix}$$ This forms a plane in $\mathbb{R}^3$.

How to Find a Basis for the Column Space

A basis for the column space is the set of pivot columns from the ORIGINAL matrix.

Step 1: Write the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 1 & 2 & 2 \end{bmatrix}$$

Step 2: Row reduce

$$\mathrm{RREF}(A) = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Step 3: Go back to original matrix

Take original pivot columns:

$$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}$$

Therefore

$$\text{Col}(A) = span\left\{\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}3\\5\\2\end{bmatrix}\right\}$$

These vectors form a basis for the column space.

Row space

Sometimes, we think of matrix rows as vectors

$$\begin{bmatrix}1 & 1 & 2\\0 & 1 & 2\end{bmatrix} \Longrightarrow v_1 = \begin{bmatrix}1\\1\\2\end{bmatrix},\ \ v_2 = \begin{bmatrix}0\\1\\2\end{bmatrix}$$

When you hear the term row space

The space span by the rows of a matrix.

Suppose

$$A = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_m \end{bmatrix}$$

where $r_1, r_2, \dots, r_m$ are the rows of A.

The **row space** of A is the set of all linear combinations of the rows.

$$\text{Row}(A) = span\{r_1, r_2, \dots, r_m\}$$

or equivalently

Row(A)={$c_1\ r_1\ + c_2\ r_2$ +⋯+$c_m\ r_m$:ci∈R}

Compare Column Space and Row Space

For

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

Column Space

Columns are

$$\begin{bmatrix}1\\4\end{bmatrix}, \begin{bmatrix}2\\5\end{bmatrix}, \begin{bmatrix}3\\6\end{bmatrix}$$

So

$$\text{Col}(A) = span\left\{\begin{bmatrix}1\\4\end{bmatrix}, \begin{bmatrix}2\\5\end{bmatrix}, \begin{bmatrix}3\\6\end{bmatrix}\right\}$$

This is a subspace of $\mathbb{R}^2$.

Row Space

Rows are

$$\begin{bmatrix}1 & 2 & 3\end{bmatrix}$$

and

$$\begin{bmatrix}4 & 5 & 6\end{bmatrix}$$

Thus

$$\text{Row}(A) = span\{\begin{bmatrix}1 & 2 & 3\end{bmatrix}, \begin{bmatrix}4 & 5 & 6\end{bmatrix}\}$$

This is a subspace of $\mathbb{R}^3$.

Example

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Rows:

$$r_1 = [1\ 0\ 1]$$
$$r_2 = [0\ 1\ 1]$$

Row space:

$$\text{Row}(A) = \mathit{span}\{[1\ 0\ 1], [0\ 1\ 1]\}$$

Any vector in the row space is

$$c_1[1\ 0\ 1] + c_2[0\ 1\ 1]$$

Thus every vector in the row space has this form.

Null space

Given a matrix $A \in R^d$, the null space of A or

$Null(A) = \{x: x \in R^d \ and \ Ax = 0\}$

For example $x_1 + 2x_2$=0

$2x_1 + 4x_2$=0

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -4 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Therefore

$$C \begin{bmatrix} -2 \\ 1 \end{bmatrix} \in Null(A)$$

Every single point that will give you a 0 is in this space

We can solve for the null space by solving the augmented matrix with 0 as y

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Longrightarrow x_1 + 2x_2 = 0$$

$$\Longrightarrow x_1 = -2x_2$$

$$\Longrightarrow \ Nall(A) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathrm{C} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Example

$$Ax = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Null of of A = All solution

$$Ax = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} c \\ c \\ -c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ the solution of this system } \begin{bmatrix} c \\ c \\ -c \end{bmatrix}$$

So the null (A)= $C\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

Check that the solution to $\boldsymbol{Ax = 0}$ give a subspace ?

If x_1 and $x_2$are the solutions of $Ax = 0$

The Ax_1=0 and Ax_2=0 then A($x_1 + x_2$)=0

By distributive low we can write A($x_1 + x_2$)=0

1) Ax_1 +Ax_2=0
2) A(cx_1)=0

Once again

Given a set of vectors v_1, v_2 ... and we let these vectors be the columns of a matrix $A = [v_1 \;\; v_2 \;\; v_3 \ldots]$.

Definition: the null space is the set of all possible solutions for x such that

$$x_1 v_1 + x_2 v_2 + \cdots = 0 \quad \text{or } Ax = 0$$

To find the null space we simply create the augmented matrix such that

$$[A|0]$$

The point of this augmented matrix is that we are trying to find the solution x that will give

you the output of 0 (this is the definition of null space)

For example, the matrix A would be converted to augmented matrix as

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

We turn it into the RREF and it gives us the solution for x

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases}$$

This is an example of trivial solution when $x = 0$

Because obviously

$$A0 = 0$$

However, once we have reduced the augmented matrix down to RREF, you might get a non-trivial solution

$$\underbrace{\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]}_{\text{RREF}} \xrightarrow{to\ equations} \underbrace{\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\text{Infinite solutions}} \xrightarrow{implying} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}}_{\text{non-trivial solutions}}$$

x_2 is free variable

Solution here is a space defined by the vector, hence ,
non-trivial solution.

If we have 2 vectors it can at most span 2d

Given 2 vectors

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

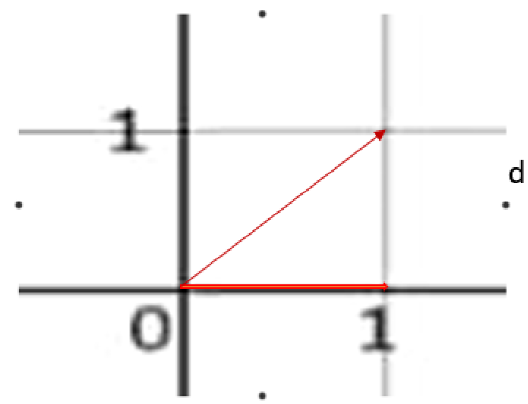
Their span would cover the entire 2D space

For example, the point d is in the span because

$$d = v_1 + {}^1\!/_2\, v_2$$

$$\begin{bmatrix} 1.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

The vectors used to create the span of a space are called base



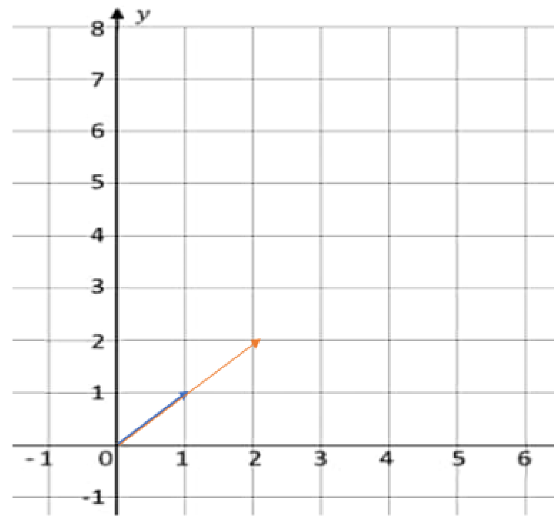
Every single point on this 2D plane can be achieved by some combination of the 2 vectors

2 vector does not imply 2d span

Given 2 vectors

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

Their span would cover only 1 dimension



The dimension span by a set of vectors is called the

Rank

This lead to have two possible cases

The rank is same as the number of basis

This happen when each vector is contributing to the new dimension.

The rank is fever than the number of basis

This happen when we have redundant vectors covering the same space.

These two cases have names

The rank is same as the number of basis

These vectors are called **linearly independent.**

The rank is fever than the number of vectors

[having redundant vectors]

These vectors are called **linearly dependent.**

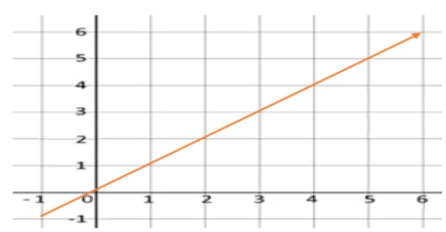
Linearly dependent and independence

Given 2 vectors

$$v_1 = \begin{bmatrix}1\\1\end{bmatrix}, \qquad v_2 = \begin{bmatrix}2\\2\end{bmatrix}$$

Their span would cover only 1 dimension

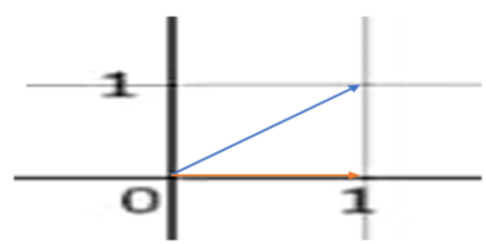
Linearly dependent



Given 2 vectors

$$v_1 = \begin{bmatrix}1\\0\end{bmatrix} v_2 = \begin{bmatrix}1\\1\end{bmatrix}$$

Linearly independent



Their span would cover the entire 2D space