Linear Algebra

Lecture 09

Vector spaces, Row spaces, Column spaces.

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Objectives

Vector Spaces

- Understand the concept of vector spaces and their properties.
- Learn vector addition and scalar multiplication.
- Identify subspaces, span, basis, and dimension.

Row Spaces

- Understand the row space of a matrix.
- Analyze relationships among matrix rows.
- Use row operations to determine rank and linear dependence.

Column Spaces

- Understand the column space of a matrix.
- Analyze how columns generate vectors through linear combinations.
- Determine whether a system of equations has a solution.

Outcomes

Vector Spaces

- Apply vector space properties to solve problems.
- Determine basis, dimension, and span of a vector space.
- Test vectors for linear independence.

Row Spaces

- Find the row space of a matrix.
- Determine the rank of a matrix.
- Identify dependent and independent equations.

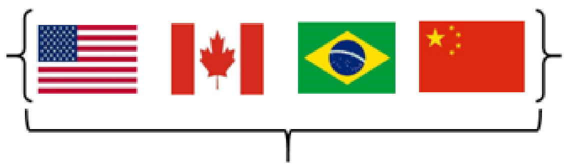
Column Spaces

- Find the column space of a matrix.
- Determine whether a vector belongs to the column space.
- Analyze the solvability of systems of linear equations.

Set: A set is a collection of distinct elements.

Examples

$$\underbrace{\{\mathrm{dog},\ \mathrm{cat},\ fish,\ bird\}}$$ Set of animals



A set of flags

$$\{1,2,3,4,5\ \ldots\}$$ Set of numbers

If nothing is inside the set it is an empty set

$$\emptyset = \{\ \}$$

A set is denoted by a cursive capital letter $\quad \mathrm{X} = \{\mathrm{a}, \mathrm{b}, \mathrm{c}\}$

$$\chi, \mathrm{y}, \mathrm{S}$$

A space is just a set with some defined properties

Just like vector sets by itself does not mean anything and we can not do much with it. For example, how do you compute {  +  }

We do not know because it was not previously defined

This is where the concept of space come in

Space is a set where certain structures are defined

For example {  +  }

This could mean the summation of the population of the 2 country

vector space

vector space is just a collection of objects (which we call vectors) where you can do two main things without breaking the rules of the space:

1. **Add** any two vectors together to get a new vector that is still in the space.
2. **Scale** any vector by multiplying it by a number (a scalar) to get a new vector that is still in the space.

The 8 Rules of the vector space

If a set of objects can't pass all eight of these tests under addition and scalar multiplication, it is not a vector space.

Vector Addition Rules

These rules ensure that adding vectors together behaves predictably.

- **Commutativity:** The order doesn't matter.

If u, v are vectors

$$u + v = v + u$$

- **Associativity:** Grouping doesn't matter when adding three or more.

$$(u + v) + w = u + (v + w)$$

- **Identity Element:** There is a unique "zero vector" 0 that leaves any vector unchanged.

$$u + 0 = u$$

- **Inverse Element:** Every vector has an exact opposite that pulls it back to zero.

$$u + (-u) = 0$$

Scalar Multiplication Rules

- **Distributivity over Scalar Addition:** Scaling a vector by two added numbers is the same as scaling it by each separately and adding the results.

$$(c + d)u = cu + du$$

- **Compatibility of Multiplication:** Scaling a vector consecutively works exactly like multiplying the scalars together first.

$$c(du) = (cd)u$$

- **Identity Scalar:** Multiplying any vector by the number \$1\$ leaves it completely unchanged.

$$1u = u$$

EXAMPLE

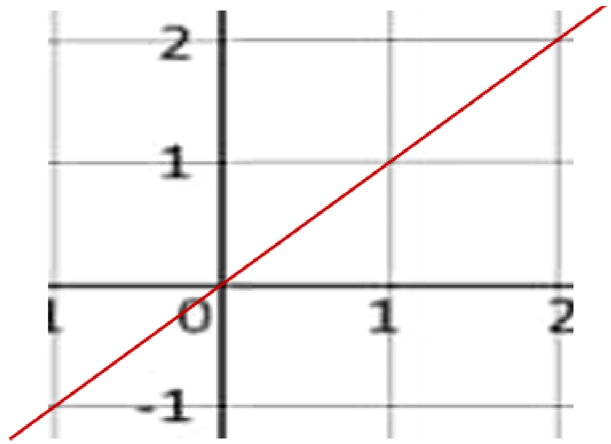
The 2D Plane ($\mathbf{R}^2$): Think of standard x, y coordinates. If you take the arrow $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and add it to $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$, you get $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$, which is still on the flat plane. If you multiply $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ by 3, you $\begin{bmatrix} 3 \\ 6 \end{bmatrix}$ still on the plane.

A set can be defined by a vector

Given the vector $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ we can define a space as all possible points reachable by some constant multiplying the vector.

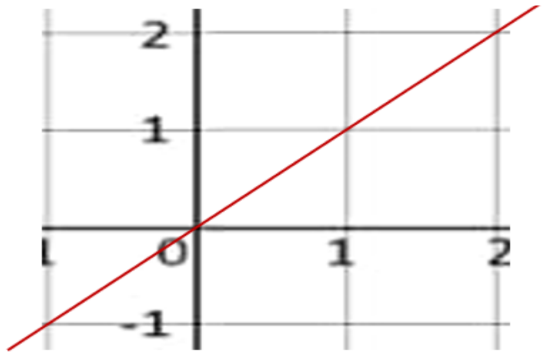
$$v = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

for example, $\quad v = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad or \quad v = -2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$



Since this set is defined by a vector, so it come with all properties of vector addition subtraction, hence it is a space with defined operations.

You can imagine every point on this line as part of the set.



This set has infinite points on it

This is an example of 2-d space

linear combination of the vectors

DEFINITION

If w is a vector in R^n, then w is said to be a linear combination of the vectors v_1, v_2 ,….. v_r in R^n if it can be expressed in the form

w = k_1 v_1 + k_2 v_2 +…+ k_r v_r where k_1, k_2 ,…, k_r are scalars.

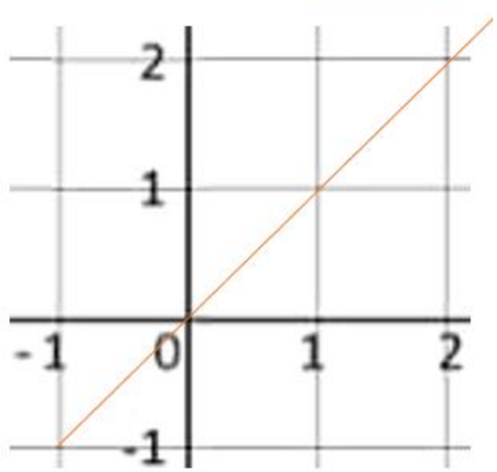
These scalars are called the coefficients of the linear combination.

Span

This space is the span of vector $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Span is very important concept

- This is the set of all possible points reachable by one or multiple scaled vectors



the span of $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

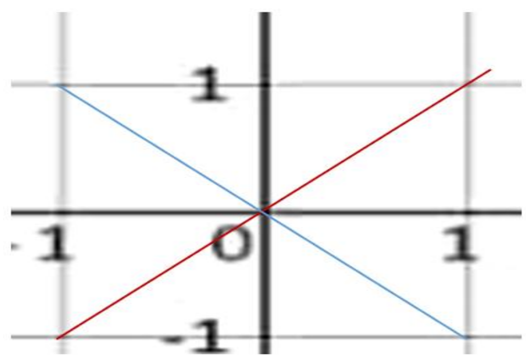
$$\begin{bmatrix}0.5\\0.5\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1.5\\1.5\end{bmatrix}, \begin{bmatrix}2\\2\end{bmatrix}, \begin{bmatrix}5\\5\end{bmatrix}, \begin{bmatrix}100\\100\end{bmatrix} \dots \dots, \begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}-0.5\\-0.5\end{bmatrix}, \begin{bmatrix}-1\\-1\end{bmatrix}, \begin{bmatrix}-2\\-2\end{bmatrix} \dots\dots\dots\dots$$

All these vectors are the same because they are all along the same line.

The red line is the span of $x = \begin{bmatrix}1\\1\end{bmatrix}$

The blue line is the span of $\mathrm{x} = \begin{bmatrix}-1\\1\end{bmatrix}$

they both intersect at point (0,0)



Span is a very important concept

it is the set all point reachable by one or multiple scaled vectors

The span of multiple vectors

With a vector v its span is

$\boldsymbol{S} = \mathrm{span}(v) = \{\alpha v : \alpha \in \mathbb{R}\}$ This notation is called the set builder notation.

It defines a set such that α can be any real number.

You may sometime see a line instead.

But you can also have multiple vectors

$v_1, v_2, v_3, \ldots,$ its span would be

$$S = \mathrm{span}(v_1, v_2, v_3 \ldots,) = \{(av_1, bv_2 \ldots): a, b \in \mathbb{R}\}$$

For example, you have 2 vectors

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

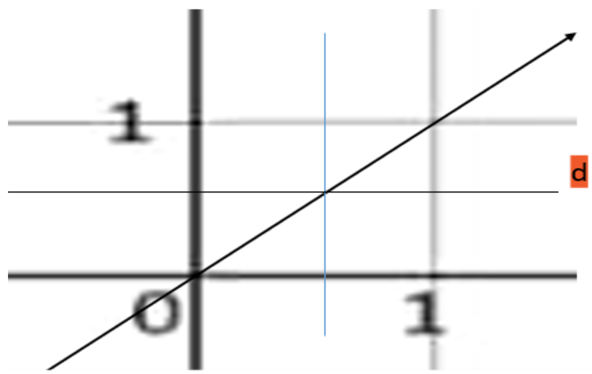
Their span would cover the entire 2D space

For example, the point d is in the span because

$$d = v_1 + {}^{1}/_{2}\, v_2$$

The vectors used to create the span of a space are called basis

$$\begin{bmatrix} 1.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$



Every single point on this 2D plane can be achieved by some combination of the 2 vectors

How do you know if a point is in the span for a set of vectors?

By definition if a set of vectors can scale and sum up to a target point, then the target is in the span

Given $\{v_1, v_2, v_3\}$ and the target vector u

$$u \in \operatorname{span}\{v_1, v_2, v_3\}\,?$$

This is equivalent to asking if there exists

a, b, c such that

$$av_1 + bv_2 + cv_2 = u$$

For example

$$a\begin{bmatrix}1\\0\end{bmatrix}+b\begin{bmatrix}1\\1\end{bmatrix}+c\begin{bmatrix}2\\2\end{bmatrix}=\begin{bmatrix}2\\1\end{bmatrix}$$

This gets harder if we have a lot of vectors

Matrix multiplication is the same operation

$$a\begin{bmatrix}1\\0\end{bmatrix}+b\begin{bmatrix}1\\1\end{bmatrix}+c\begin{bmatrix}2\\2\end{bmatrix}=\begin{bmatrix}2\\1\end{bmatrix}$$

$$\begin{cases}a+b+2c=2\\ \quad b+2c=1\end{cases}$$

$$\begin{bmatrix}1 & 1 & 2\\0 & 1 & 2\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}2\\1\end{bmatrix}$$

$$\begin{cases}a+b+2c=2\\ \quad b+2c=1\end{cases}$$

Identifying if a point is in the span of a set of vectors is identical to solving a system of linear equations.

If a solution exist (infinite or unique), then the target is in the span

This is why gaussian elimination is so important

Matric multiplication is the same operation

$$a\begin{bmatrix}1\\0\end{bmatrix}+b\begin{bmatrix}1\\1\end{bmatrix}+c\begin{bmatrix}2\\2\end{bmatrix}=\begin{bmatrix}2\\1\end{bmatrix}$$

$$\begin{bmatrix}1 & 1 & 2\\0 & 1 & 2\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}2\\1\end{bmatrix}$$

$$\begin{cases}a+b+2c=2\\ \quad b+2c=1\end{cases}$$

$$\begin{cases}a+b+2c=2\\ \quad b+2c=1\end{cases}$$

Here, we are interpreting each column as a vector

The space span by these columns vectors is called column space